Deligne's solution to the Hilbert 21^{st} problem

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The Riemann-Hilbert Correspondence builds a bridge between algebraic and analytic structures on a variety. In this talk we explain Deligne's solution to the Hilbert 21^{st} problem.

1 I: Algebraic vs analytic structure

We have an "analytification" functor that takes an algebraic variety X (defined over $\mathbb{C}$ with the Zariski topology) to an analytic variety X^{an} (viewed as a complex manifold with the analytic topology). For example, $(\mathbb{A}^1)^{\text{an}} = \mathbb{C}$ and analytification of affine varietes correspond to Stein manifolds. Analytification works similarly for coherent sheaves.

Theorem 1 (Serre's GAGA, 1956)**.** If X is a smooth proper variety, then the analytification functor induces an equivalence of categories between coherent algebraic sheaves on X and coherent analytic sheaves on X^{an}. Moreover this equivalence preserves cohomology.

Corollary 1. For smooth compact algebriac variety X:

$$H^i(X, \Omega^\bullet_{alg}) \overset{GAGA}{=} H^i(X^{an}, \Omega^\bullet_{hol}) \overset{Hol\ Poincare\ lemma}{=} H^i(X^{an}, \mathbb{C})$$

in other words, singular cohomology of $X(\mathbb{C})$ can be computed from algebraic data.

Corollary 2. There is an equivalence between

1. Algebraic vector bundles with integrable connections
2. Analytic vector bundles with integrable connections

Frobenius existence theorem in ODE says on a complex manifold X^{an}, an analytic vector bundle with integrable connection is the same as a locally constant sheaf (a representation of $\pi_1(X^{an})$) given by the flat sections of the connection. This implies the classical Riemann-Hilbert correspondence; equivalence of algebraic flat connections to representations of the fundamental group:

Corollary 3. (Riemann-Hilbert for smooth proper algebraic varieties) There is an equivalence between

1. Algebraic data: algebraic vector bundles with integrable connections on X
2. Topological data: representations of $\pi_1(X^{an})$

The representation of the flat connection is called the **monodromy** of the connection.

A key question arises: what if X is smooth but not proper (e.g., not compact)? Deligne proved in 1970 that a version of this correspondence still holds.

II: The non proper case

Deligne's strategy for a smooth, non-proper variety X is to use a compactification $\overline{X}$. The core problem becomes how to properly extend an analytic vector bundle with a connection, $(M^{\mathrm{an}}, \nabla^{\mathrm{an}})$, from X^{an} to the compactification $\overline{X}^{\mathrm{an}}$. If such an extension $(\overline{M}^{\mathrm{an}}, \overline{\nabla}^{\mathrm{an}})$ can be found, GAGA can be applied on $\overline{X}$ to get an algebraic object $(\overline{M}, \overline{\nabla})$. Then if we restrict $(\overline{M}, \overline{\nabla})$ back to X we will then have an algebraic flat bundle associated to $(M^{\mathrm{an}}, \nabla^{\mathrm{an}})$.

The local model for this problem is studying a connection on the punctured disk $U = \Delta^* = \{z \in \mathbb{C} : 0 < |z| < 1\}$, with compactification $X = \Delta$. An integrable connection ∇ on a locally free sheaf M over U is a map $\nabla : M \to M \otimes \Omega^1_U$ satisfying the Leibniz rule:

$$\nabla(fm) = f\nabla m + m \otimes df$$

For $M = (\mathcal{O}_U)^n$, a connection can be written locally as $\nabla(f) = df + A(z)f dz$, where $A(z)$ is a matrix of holomorphic functions on U.

The existence and uniqueness theorem for ordinary differential equations allows for analytic continuation of local solutions. Continuing a basis of solutions around a loop in Δ^* induces a linear automorphism of the fiber, giving rise to the monodromy representation:

$$\rho : \pi_1(\Delta^*) \to \mathrm{Aut}(V_{z_0})$$

where V_{z_0} is the stalk of the sheaf of horizontal sections at a point $z_0 \in \Delta^*$.

Example: Consider the equation $\frac{df}{dz} = \frac{\alpha f}{z}$ for $\alpha \in \mathbb{C}$. The solution is $f(z) = z^\alpha = e^{\alpha \log z}$. As we circle the origin once, $\log z \mapsto \log z + 2\pi i$, so the solution transforms as $f(z) \mapsto e^{\alpha(\log z + 2\pi i)} = e^{2\pi i \alpha} f(z)$. The monodromy is multiplication by $e^{2\pi i \alpha}$.

Regular Singularities

The notion of a "good" singularity is crucial for the correspondence.

Definition 1 (Regular Connection)**.** A differential equation is said to have a regular singularity at $z = 0$ if its local solutions on Δ^* have "moderate growth", i.e., for any solution $f(z)$, there exists an integer $N \geq 0$ such that $|f(z)| = O(|z|^{-N})$ as $z \to 0$.

Fuchs (1860s) provided an algebraic equivalent to this analytic condition! A differential operator $P = \partial_z^n + a_1(z)\partial_z^{n-1} + \cdots + a_n(z)$ has a regular singularity at $z = 0$ if and only if the coefficients $a_i(z)$, which are meromorphic functions, have poles of at most order i at $z = 0$. That is, $\mathrm{ord}_{z=0}(a_i) \geq -i$.

This allows the concept of regularity to be defined algebraically, which is necessary for the correspondence. Deligne's main result is that the Riemann-Hilbert correspondence holds for any smooth algebraic variety U if we restrict to connections with regular singularities.

Problem 1. (Hilbert 21^{st} problem) Let X be a smooth projective curve and $U = X \backslash \{points\}$. U^{an} is a punctured Riemann surface. There should be an equivalence between

1. Algebriac connections with regular singularities on U
2. Local systems on U^{an}

Delgine proves this for any smooth algebraic variety U.

Deligne's Extension Theorem

The key to Deligne's proof is the canonical extension of a regular connection. Deligne's method works for normal crossing divisors but for simplicity we focus on the local model Δ^*.

Definition 2 (Regular Meromorphic Connection)**.** A connection (M, ∇) on a smooth variety X with respect to a divisor D is a regular meromorphic connection if it has at most logarithmic poles along D. This means that in local coordinates, the connection matrix $A(z)$ can be written as $A(z) = B(z)\frac{dz}{z}$, where $B(z)$ is holomorphic.

Theorem 2 (Deligne, 1970)**.** Let (M, ∇) be a vector bundle with a flat, integrable connection on $U = \Delta^*$. Then there exists a unique extension of (M, ∇) to a regular meromorphic connection $(\tilde{M}, \tilde{\nabla})$ on $X = \Delta$, such that

1. $\tilde{M} = O_X(*) \otimes_{O_X} L$ for a locally free O_X module L
2. $\tilde{\nabla} : \tilde{M} \to \Omega^1_X \otimes \tilde{M}$
3. the eigenvalues α of the residue of $\tilde{\nabla}$ at $z = 0$ satisfy $\mathrm{Re}(\alpha) \in [-1, 0)$.
4. $(\tilde{M})^{\tilde{\nabla}}_U = M^{\nabla}$.
5. The map $(M, \nabla) \mapsto (\tilde{M}, \tilde{\nabla})$ is an equivalence.

Proof. we sketch the existence and the hard part is the uniqueness. Let C be the image of the monodromy map of $\pi_1(\Delta^*)$ in $Gl_n(\mathbb{C})$. By linear algebra there exists a unique $T \in Mat_{n\times n}(\mathbb{C})$ such that

$$e^{2\pi_i T} = C$$

such that the real eigenvalues of T are in $[-1, 0)$. Let $L = O_X^{\oplus n}, \tilde{M} = O_X(*)^{\oplus n}$ and define

$$\tilde{\nabla}(e_i) = \sum_j T_{i,j} \frac{dz}{z} \otimes e_j$$

□

Corollary 4. (Deligne's proof of Hilbert 21^{st} problem)

1. $\{\pi_1(\Delta^*)\} = \{(M, \nabla) \in Conn(\Delta^*)\}$
2. $\{(M, \nabla) \in Conn(\Delta^*)\} = \{(\tilde{M}, \tilde{\nabla}) | \text{regular-meromorphic connection on } \Delta\}$
3. By GAGA $(\tilde{M}, \tilde{\nabla})$ is algebraic.
4. By restricting to Δ^* we get an algebraic object corresponding to $\{\pi_1(\Delta^*)\}$.

Generalization

The upshot of Deligne's work is an equivalence of categories for any smooth compact algebraic variety X and a simple normal crossing divisor D:

$$\left\{ \begin{array}{c} \text{Regular-meromorphic connections} \\ \text{on } X \text{ with poles along } D \end{array} \right\} \cong \left\{ \begin{array}{c} \text{Local systems (representations of } \pi_1) \\ \text{on } U = X \setminus D \end{array} \right\}$$

This correspondence has been vastly generalized using the language of D-modules. A differential equation can be rephrased as a module over the ring of differential operators, D_X. The space of solutions is then given by $\mathrm{Hom}_{D_X}(\mathcal{M}, \mathcal{O}_X)$. i.e., given a differential operator P define the D module $M = D_X/PD_X$. Then solutions of $\{Pf = 0\}$ is the same as $\{Hom_{D_X}(M, O_X)\}$. This is the Sol functor.

Theorem 3 (Kashiwara 1980s)**.** For a complex manifold X, the Sol functor

$$Sol : D^b_{rh}(D_X) \to D^b_c(\mathbb{C}_X)^{op}$$

defines an equivalence of categories.

By dualizing we get the deRham functor

$$DR : D^b_{rh}(D_X) \to D^b_c(\mathbb{C}_X)$$

$$DR(M) = Sol(DM)[n] = \Omega_X \otimes_{D_X} M$$

The hard part is the proof of constructability of the sheaf of solutions $Sol(M)$ of the differential equation corresponding to M.