Arakelov Geometry

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1 History

The references are Soulé's "Lectures on Arakelov Geometry" and Ziyang ZHU's "A Concise Course in Arakelov Geometry".

This talk is about a very quick introduction to Arakelov geometry motivated by some low dimentional cases. First a little about the history:

1. dim$= 1, X = Spec(\mathcal{O}_K)$ for a number field K, essentially goes back to André Weil, the classical analogy between a number field and a function field.
2. (1974) dim=2, X= surface/$\mathcal{O}_K$, this is the original foundation of Arakelov theory developed by Arakelov in a 1974 paper about intersection of divisors on X.
3. (1984) "Calculus on Arithmetic surface" by Faltings where he proved Riemann-Roth, Hodge index theorem among other things. His main tool was Theta divisors on Jacobians of curves.
4. (1985-1992) Arakelov geometry in higher dimension using Currents developed by Gillet and Soulé. The general Gorthendieck-Riemann-Roth were obtained using analytic results of Bismut.
5. 1991 Mordel-Lang conjectures proved using Arakelov geomtry by Vojta and Faltings.
6. Recent work by Shou-Wu Zhang.

2 Dim 1, $X = Spec(\mathcal{O}_K)$

There are 3 equivalent ways to study primes of a ring of integer $\mathcal{O}_K$. All of the following SETS are the same,

1. $Cl(\mathcal{O}_K) = K^* \setminus \mathbb{A}_K^*/\mathcal{O}_K^* K_\infty^*$: this is fractional ideals module principal ideals
2. $Pic(Spec(\mathcal{O}_K))$: this the group of locally free rank one (line bundles) O_X-modules.
3. $CH^1(\mathcal{O}_K)$= this is the first Chow group; divisors module principal divisors

If you haven't seen this it is a good exercise to come up with the morphisms between them.

The idea is now to compactify these spaces by adding information at ∞ places. They become

1. $\widehat{Cl(\mathcal{O}_K)} = K * \setminus \mathbb{A}_K^*/\mathcal{O}_K^* \mathcal{O}_\infty^*$
2. $Pic(\widehat{Spec(\mathcal{O}_K)})$: this consists of metrized line bundles, i.e., $(L, ||.||)$ where $||.||$ is a metric on $L \otimes \mathbb{C}$, a familiy of Hermitian inner products on fibers.
3. $\widehat{CH^1(\mathcal{O}_K)} = \{div(X) \oplus (\bigoplus_{\tau \in Hom(K,\mathbb{C})} \mathbb{R})^{Gal(\mathbb{C}/\mathbb{R})}\}/\{\Sigma\, ord_p(x)p, -log|\tau(x)|^2\}$

Let me describe the isomorphisms between these 3 sets. Given some $(L, ||.||) \in Pic(\widehat{Spec(\mathcal{O}_K)})$, we get an element in $\widehat{Cl(\mathcal{O}_K)}$ by

$$(L, ||.||) \mapsto ((\overline{w}_p)^{n_p}, ||s_\tau||_\tau)$$

where $\overline{w}_p$ is a uniformizer and $s \in H^0(L)$ is a global section.

We get an element in $\widehat{CH^1(\mathcal{O}_K)}$ by

$$(L, ||.||) \mapsto \widehat{div(s)} = (\sum n_p p, -log||s_\tau||_\tau^2)$$

n_p is the order of vanishing of s at p.

2.1 Arithemtic degree

We can say a lot more than just equality of these 3 sets. For example we can prove the Arithemtic-Riemann-Roth here. To do that we need the notion of Arithemtic degree of $(L, ||.||)$,

$$\widehat{deg}((L, ||.||_\tau)) = log\,\#(\frac{L}{s.\mathcal{O}_K}) - \Sigma_{\tau \in Hom(K,\mathbb{C})} log||s_\tau||_\tau$$

or equivalently

$$(\Sigma\, n_p, n_\infty) \mapsto \Sigma\, n_p log\, p + \frac{n_\infty}{2}$$

By the product formula, the sum is independent of s.

2.2 Euler characteristic

Define

$$\hat{\chi} : \widehat{\mathrm{Pic}}(X) \to \mathbb{R}$$

$$(\mathcal{L}, \|\cdot\|) = (\Sigma\, n_p, r_\tau) \mapsto -\log(\mathrm{Vol}(lattice\ \frac{K_\mathbb{R}}{\rho_L L \subset K_\mathbb{R}})$$

where

$$\rho_L : K_\mathbb{R} \to K_\mathbb{R}$$
$$(s_\tau) \mapsto e^{\frac{r_\tau}{r}}.s_\tau$$

For example the Minkowski theorem implies

$$\hat{\chi}(\mathcal{O}_K) = -log\sqrt{disc(K/\mathbb{Q})}$$

2.3 Arithmetic Riemann-Roch:

$$\hat{\chi}(L, ||.||_\tau) - \hat{\chi}(\mathcal{O}_K) = \widehat{deg}(L, ||.||_\tau)$$

3 Dim 2, surfaces, $X/\mathcal{O}_K$

This is what Arakelov did: the idea is to see X as fibered over finite primes, the fiber over ∞ is a Riemann Surface. A generic point P determines a point P on the Riemann Surface X_∞ The first thing we need is the notion of a metrized line bundle. Given an Arakelov divisor $(D_{finite}, \sum_{infinite\,\sigma} r_\sigma P_\sigma)$ we take the metrized line bundle

$$(\mathcal{O}_K(D), a\,canonical\,metric\ \times\ e^{-r_\sigma})$$

This canonical metric comes from Green function g s.t. for $1_P = \in H^0(\mathcal{O}(P)$

$$||1_p||(Q) = g(P, Q)$$

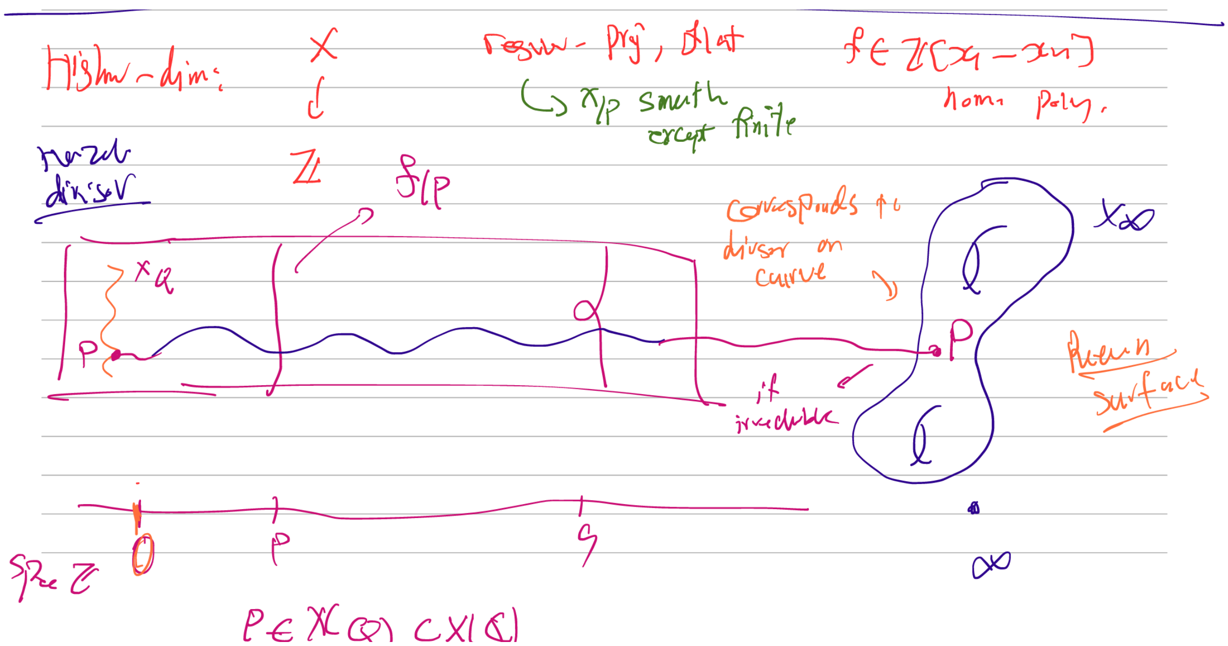

Figure 1: X fibered over primes

I want to motivate where this g comes from. It comes from the study of intersection theory of divisors. Consider two finite divisors D, E, E irreducible horizontal, define the finite intersection as

$$(D, E)_{fin} = \sum_{v \in \mathcal{O}_K} (D, E)_v$$

$$(D, E)_v = \sum_{x|v} i_x(D, E) log(\#k(x))$$

the sum over each fiber v, and intersection on each fiber is a sum over primes over v with intersection number i_x multiplied by the residue field.

Fact: if f is a local defining function of D, $D = div(f)$

$$(D, E)_v = \sum -log||f|_E||_{p_i}$$

By analogy we want to define the intersection at infinty by

$$(D, E)_\infty = \sum (P_\infty, Q_\infty)_{\alpha,\beta}$$

with

$$(P_\infty, Q_\infty)_{\alpha,\beta} = -log\phi_P(Q)$$

Here D, E correspond to P, Q on generic fiber with residue fields $L_P, L_Q \hookrightarrow \mathbb{C}$ embedded by various $\infty_\alpha, \infty_\beta$. P_∞, Q_∞ are points on a Riamann Surface X_∞. The problem is how to find the functions ϕ_P. Since by symmetry we need to have

$$\phi_P(Q) = \phi_Q(P)$$

we need to find a good function

$$\phi : X_\infty \times X_\infty \to \mathbb{R}$$

with this property. We need a family of functions $\{\phi_P\}$. The idea which goes back to Parchin is to have $\{\phi_P\}$ as solutions to differntial equations and use the fact that Harmonic functions are constant on a compact Riemannian manifold. The differential equation is

$$\frac{1}{2\pi}\Delta log \phi_P = -d\mu$$

Now

$$\phi_P - \phi_Q = constant$$

as they're Harmonic. The set of solutions are given by the Green function g with

$$\partial\overline{\partial}g(P,z) = -2\pi\mu$$

for $z \neq P$. Then we take

$$\phi_P(-) = g(P,-)$$

4 Higher dimension and currents

It's not quite clear why the Green function works perfectly in the surface case and how to generalize this notion in higher dimensions. Also the importance of the differential equation $\partial\overline{\partial}g(P,z) = -2\pi\mu$ is vague. To understand this we need to know a little about currents. A good reference is one of the chapters of Griffiths-Harris book. This is what unifies both differential forms and cycles. Remember the Poincaré duality where it says differential forms and topological cycles are essentially duals. You can obtain a distribution on the space of smooth differential forms A, a **current**, from both a cycle and a differential form:

Given a cycle γ define the functional on the space of differential forms

$$\delta_\gamma : \alpha \mapsto \int_\gamma \alpha$$

Given a differential form β define the functional

$$\alpha \mapsto \alpha \wedge \beta$$

Hence we define level n currents as $D_n = (A^n)^*$ where the elements are distributions. Differential operators like ∂ induce differentials on these distributions and we have similar decompositions into p,q forms.

The generalization of $\partial\overline{\partial}g(P,z) = -2\pi\mu$ becomes the definition for a Green-current for a codim p analytic subvariety Y: We say $g \in D^{(p-1,p-1)}(X)$ is such a thing if

$$dd^c g + \delta_Y = [w]$$

where w is a smooth honest $A^{(p,p)}$ differential form. An important example for principal divisors $Y = \mathrm{div}(f)$ comes from the **Poincaré–Lelong** formula where it says if L is a holomorphic line bundle on a complex manifold and $||\cdot||$ is a Hermitian metric on it, and c_1 the first Chern form, then

$$dd^c(-\log ||s||^2) + \delta_{\mathrm{div}(s)} = [c_1(L, ||\cdot||)]$$

Therefore $-\log ||s||^2$ is a current for the principal divisor, this is a similar object showing up in Green functions for surfaces.

The reason that only one function $g(P,Q)$ shows up in the whole story of surfaces is that in codimension 1, the only nontrivial case for surfaces, the Green function packages all the Green-currents for $\mathcal{O}_X(D)$ into one function g. In other words, $g(P,z)$ gives all the currents. Therefore the Green function fixes a system of currents for all the divisors. In higher codimensions, this is no longer true that we can package all the currents into one object, so we have to consider all of them.

For example the definition of Arakelov cycles $\widehat{Z^k}$ becomes pairs of (Z, g_Z) where g_Z is a Green-current for cycle Z. Qutienting by principal Arakelov divisors defines $\widehat{CH^k}(X)$. There is a notion of "product" of currents which can be used to make the chow group into a ring. One can develop a theory of Chern classes for Arakelov vector bundles, i.e., a vector bundle E/X on X with a Hermitian metric on $E(\mathbb{C})/X(\mathbb{C})$. The crown jewel is the Arithmetic-Grothendieck–Riemann–Roch theorem. Two importnat inputs among other things are the Mumford notion of determinant of cohomology and Quillen metrics.