GAGA Correspondence

Mohammadali Aligholi

November, 2021

Contents

1 Introduction

Algebraic geometry is a meeting place where many fields of mathematics converge. Commutative algebra, Complex analysis, and Geometry are three quite distinct subjects which somehow magically related to each other in algebraic geometry. We want to understand one of the most important articles in 20-th century algebraic geometry by Serre that brings together many fields of mathematics.

Originally, GAGA refers to the celebrated paper *Géometrie Algébrique et Géométrie Analytique* by Serre in 1956. Serre showed a tight relationship between complex algebraic geometry and complex geometry by proving three theorems, known as GAGA theorems. Today, GAGA is used for any theorem of comparison, allowing passage between algebraic geometrical categories to the well-defined analytic geometrical categories.

The goal of this paper is to give a detailed exposition of classical GAGA theorems and analogues theorems in Formal geometry, namely the Comparison theorem and the Existence theorem of Grothendieck. In the end we state and prove a special case of a unified GAGA theorem due to J. Hall, which encompass all the versions of GAGA theorem.

1.1 The basic idea

There are two central objects in geometry, one the notion of Manifold which can be real or complex and looks locally as open sets of $\mathbb{R}^n$ or $\mathbb{C}^n$, and the other is the notion of variety. An affine variety is the common zeros of a systems of polynomials that can be defined in any field, and a variety is a geometric object which looks locally as an affine variety in the same sense as a manifold. Regular levels sets of smooth and holomorphic functions are prime examples of Manifold, particularly level sets of polynomials. As a result, if we work in the field of $\mathbb{C}$ then these two geometric objects are related to each other specially if we consider *analytic spaces* which allows manifolds to have singularity. But as topological spaces they have a very different topologies.

Because of rich theory of complex analysis and also the fact that $\mathbb{C}$ is an algebraicly closed field, a complex variety is an excellent object to study if we want to know more about the connections of these two geometric objects. Also if we have a complex variety it can be studied from two points of view; an *algebriac* view where we can consider the space of regular functions on it, and an *analytic* view where we can study the space of holomorphic functions on it. If our variety is non-singular then we have the full machinery of theory of Kahler manifolds. One of the major result concerning these two points of view is Chow's Theorem which says that a closed analytic subspace of a complex projective space is in fact an algebraic variety. In 1956, Serre consolidates the relationships between these two views and unified the same results we have, although by very different methods. Before we state Serre's theorems we should say a bit about sheaves and cohomology.

Using sheaves as an effective tool for studying geometric objects started around mid 20-century in complex analysis. A sheaf simply assigns an algebraic structure such as a ring, to every open sets of a topological space. For example the ring of continuous functions on each open sets. Space of regular functions and holomorphic functions discussed above are another examples. It turns out that all of the geometric objects, like manifolds and algebraic varieties, can be seen as a *ringed space*, a topological space with a specific sheaf of rings attached to it.

Sheaves can be viewed abstractly as a functor between the category of open sets of topology to the category of desired algebraic structure. Sheaves provide the framework for a very general cohomology theory, another rich theory of past century. Cohomology is a method to assign a sequence of algebraic structures such as abelian groups to a topological space like singular cohomology in algebraic topology . Sheaf cohomology is a vast generalization of singular cohomology, allows us to define cohomology with coefficients in any sheaves not just abelian groups.

An important class of sheaves both in algebraic geometry and complex geometry is *coherent sheaves*. And its corresponding cohomology theory is very powerful. Coherent analytic sheaves first were created by Oka and Henri Cartan in the theory of Complex analysis in several variables. Soon afterward, Serre introduced coherent sheaves into algebraic geometry.

Now we can state what did Serre show in his famous paper, Géometrie Algébrique et Géométrie Analytique, known as *GAGA*. He showed that all of these general results concerning close relationships between algebraic geometry and analytic geometry, reduced to the fact that we have an equivalence of categories between the category of coherent algebraic sheaves on an algebraic object like a scheme or a variety and the category of coherent analytic sheaves on its corresponding analytic space and this correspondence leaves the cohomology groups invariant. This theorem has numerous and far reaching applications.

Although, the original theorem is a result in classical algebraic geometry concerning projective varieties, GAGA theorems can be stated more generally for schemes, like for locally of finite type schemes over $\mathbb{C}$.

There are analogues theorems of GAGA, like *formal GAGA* about *Formal schemes* and *rigid GAGA* in rigid geometry. In this article we present one of them which is Formal GAGA.

The formal scheme were motivated by and generalize the theory of formal holomorphic functions. By the notion of completion of a scheme along a subscheme we can present comparison theorems of *GAGA* type proved by Grothendieck between algebraic geometry and formal geometry.

Recently, there are attempts to unify thses various versions of GAGA theorems into a single framework. First Jack Hall proved a general GAGA theorem using Bondal representabilty theorems, It gives all the three versions of GAGA. Later, Amnon Neeman proved a more general theorem by using approximations of triangulated categories. In the last chapter we prove a special case of J. Hall theorem which is enough to imply the usual GAGA theorems of Serre.

1.2 Sections

To make this paper self-contained as much as possible, the first part is dedicated to preparation materials needed to understand and prove the theorems. We need to know about sheaves and sheaf cohomology and some commutative algebra results. We need some nontrivial results in both algebraic geometry and complex analysis. Vanishing theorem of Serre and finiteness theorem are the essence of the proof of GAGA theorems. In the end, we finish this chapter by reviewing classical Weierstrass theorems in several complex analysis and the definition of analytic space.

Next, we switch to analytic theorems which are due to Cartan, Serre, Oka,... . These results are analogous to the algebraic results about schemes and varieties, such as the finiteness and vanishing theorems of Serre which we have proved in the preliminaries section. However, unlike their algebraic counterparts, their proof are much more involved. For example we prove a deep theorem of Oka. It says the sheaf of holomorphic functions is coherent. Next we present famous theorems A and B of Cartan about stein spaces. The last major theorem we need is that sheaf cohomology groups of a compact complex space are finite dimensional. In most expository papers for GAGA, these theorems are taken as black boxes. Although their proof are nontrivial, we'll present the ideas since most of the materials in algebraic geometry like coherent sheaves and sheaf cohomology were entered into algebraic geometry by people like Serre because they saw the power of these tools in complex analysis. By saying a bit about these complex analysis results, we will know more about the motivation behind techniques used in algebraic geometry.

Second, we present the GAGA paper itself. We'll start from Analytification and end with the proof of all theorems. Much of the ingredients of the proofs are the ones we have obtained in the previous sections.

Thirdly, we are going to prove Formal GAGA theorems. First we start with the introduction of formal schemes and then we prove the theorems, which are the consequences of two main theorems, namely the Comparison theorem and the Existence theorem of Grothendieck.

At last, we try to prove a unified GAGA theorem of J. Hall. In order to state the proof we need the language of derived category and triangulated category. Then, by using a representabilty theorem due to Bondal and Vand Den Bergh we prove the theorem.

1.3 Acknowledgments

I would like to deeply thank Prof. Amir Jafari for his patience and guidance during this project. I've learned many exciting things from him during my undergraduate that deepened my interests in pure mathematics. I am really grateful for all our interaction and all the fascinating classes I took with him which led me to become interested in algebraic geometry and study this amazing piece of mathematics.

2 Preliminaries

This section is dedicated to some preliminary results about sheaves, sheaf cohomology, algebraic geometry and complex analysis.

2.1 Sheaves

First, we assume the reader is familiar with the notion of a sheaf and ringed spaces. In this section we want to study a very special class of sheaves, namely the (quasi)coherent sheaves, which is of utmost importance both in algebraic geometry and complex geometry. Roughly speaking, these sheaves can be seen as the generalization of finite dimensional vector bundles. Their corresponding cohomology theory is very powerful. Coherent analytic sheaves first were created by Oka and Henri Cartan in the theory of Complex analysis in several variables. Soon afterward, Serre introduced coherent sheaves into algebraic geometry.

2.1.1 Coherent sheaves on a ringed space

Let $(X, \mathcal{O}_X)$ be a ringed space. We assume all the stalks O_x are commutative rings with unit and all sheaves are sheaves of modules. By $\mathcal{O}_X^n$ we mean the direct sum $\mathcal{O}_X \oplus \ldots \oplus \mathcal{O}_X$.

Suppose $U \subset X$ is an open set and $\mathscr{F}$ a is sheaf on X. Let $f_1, \ldots, f_n \in \mathscr{F}(U)$. We can construct a morphism of sheaves $\phi : \mathcal{O}_X^n|_U \to \mathscr{F}|_U$ generated by $f_1, \ldots, f_n$ defined by $\phi(V) : \mathcal{O}_X^n|_V \to \mathscr{F}|_V, (r_1, \ldots, r_n) \mapsto \sum r_i f_i|_V$ for open set $V \subset U$. Conversely given a morphism $\phi : \mathcal{O}_X^n|_U \to \mathscr{F}|_U$, define the sections

$$f_1 = \phi(U)(1, 0, \ldots, 0), \ldots f_n = \phi(U)(0, 0, \ldots, 1)$$

It can be easily seen that ϕ is the morphism generated by $f_1, \ldots, f_n$.

Definition 2.1.1. *We say $\mathscr{F}$ is of **finite type** over $(X, \mathcal{O}_X)$, if every point in X has an open neighborhood $U \subset X$ such that there is a surjective morphism $\phi : \mathcal{O}_X^n|_U \to \mathscr{F}|_U$ for some natural number n.*

Lemma 2.1.2. *If $\mathscr{F}$ is of finte type over $(X, \mathcal{O}_X)$, then $Supp\ \mathscr{F}$ is closed.*

Proof. Suppose $\mathscr{F}_x = 0$, by definition there exist sections $t_1, \ldots, t_k$ on an open neighborhood U of x generating $\mathscr{F}|_U$. We have $(t_i)_x = 0$, so let $V \subset U$ an open neighborhood small enough where $(t_i)|_V = 0$ for all i. Therefore, $V \cap Supp\ \mathscr{F} = \emptyset$. □

Now we can define a coherent sheaf:

Definition 2.1.3. *A **coherent sheaf** on a ringed space $(X, \mathcal{O}_X)$ is a sheaf $\mathscr{F}$ satisfying the following two properties:*
(a) $\mathscr{F}$ is of finite type,
(b) for any open set $U \subset X$, any natural number n, and any morphism $\phi : \mathcal{O}_X^n|_U \to \mathscr{F}|_U$ of $\mathcal{O}_X$-modules, the kernel of φ is of finite type.

Morphisms between coherent sheaves are the same as morphisms of sheaves of $\mathcal{O}_X$-modules.

Lemma 2.1.4. *If $\mathscr{F}$ is coherent over $(X, \mathcal{O}_X)$, for each $x \in X$ there is an open set $x \in V \subset X$ and natural numbers m, n such that there is an exact sequence*

$$\mathcal{O}_X^m|_V \to \mathcal{O}_X^n|_V \to \mathscr{F}|_V \to 0.$$

Proof. By (a) of the definition of coherence, we can find an open neighborhood $x \in U$ and a surjecive morphism $\phi : \mathcal{O}_X^n|_U \to \mathscr{F}|_U$. let Ψ be the kernel of this sheaf homomorphism. By (b) of the definition of coherent, we can find an open neighborhood V, $x \in V \subset U$ and surjective morphism $\phi : \mathcal{O}_X^m|_V \to \Psi|_V$. Thus over V we get the exact sequence $\mathcal{O}_X^m|_V \to \mathcal{O}_X^n|_V \to \mathscr{F}|_V \to 0$. □

Definition 2.1.5. *A* ***quasi-coherent*** *sheaf on a ringed space $(X, \mathcal{O}_X)$ is a sheaf $\mathscr{F}$ of $\mathcal{O}_X$-modules such that every point in X has an open neighborhood U in which there is an exact sequence*

$$\mathcal{O}_X^{\oplus I}|_V \to \mathcal{O}_X^{\oplus J}|_V \to \mathscr{F}|_V \to 0.$$

for some (possibly infinite) sets I and J.

From the above lemma 2.1.4 it is obvious that every coherent sheaf is a quasi-coherent sheaf.

It is clear that any subsheaf of finite type of a coherent sheaf is coherent. since if the condition (b) is satisfied by a sheaf $\mathscr{F}$, any subsheaf of $\mathscr{F}$ satisfies it to.

Definition 2.1.6. *The structure sheaf $\mathcal{O}_X$ is called coherent if it is coherent considered as a sheaf of modules over itself.*

There are two important examples of coherent sheaves of rings,

(1) If X is the complex analytic space, the sheaf of germs of holomorphic functions on X is a coherent sheaf of rings. This is a deep theorem in complex analysis due to K. Oka. We'll prove it in part II.

(2) If X is an algebraic variety, the sheaf of local rings of X is a coherent sheaf of rings. Also on a locally Noetherian scheme X, the structure sheaf $\mathcal{O}_X$ is a coherent sheaf of rings (2.4.20).

When $\mathcal{O}_X$ is a coherent sheaf of rings, converse of the lemma 2.1.4 holds:

Lemma 2.1.7. *If $\mathcal{O}_X$ is a coherent sheaf of rings, $\mathscr{F}$ is coherent over $(X, \mathcal{O}_X)$ if and only if for each $x \in X$ there is an open set $x \in V \subset X$ and natural numbers m, n such that there is an exact sequence*

$$\mathcal{O}_X^m|_V \to \mathcal{O}_X^n|_V \to \mathscr{F}|_V \to 0.$$

Proof. Lemma 2.1.4 gives one direction. If $\mathcal{O}_X$ is coherent, $\mathcal{O}_X^m|_V$ and $\mathcal{O}_X^n|_V$ is also coherent(corollary 2.1.9 in the next section), also by the same corollary $\mathscr{F}|_V$ is coherent(it is a cokernel). Being coherent is a local property, thus $\mathscr{F}$ is coherent. □

2.1.2 Category of Coherent Sheaves

The coherent sheaves form an abelian category, in fact is a full subcategory of the category of $\mathcal{O}_X$-modules. So the kernel, image, and cokernel of any map of coherent sheaves are coherent. The direct sum and tensor proudct of two coherent sheaves are coherent.
To prove these rasults, we state a very important property of coherent sheaves.

Theorem 2.1.8. ***(2 out of 3 property).*** *Let $0 \to \mathscr{F} \xrightarrow{\alpha} \mathscr{G} \xrightarrow{\beta} \mathscr{K} \to 0$ be an exact sequence of sheaves. if two of the sheaves $\mathscr{F}, \mathscr{G}, \mathscr{K}$ are coherent, so is the third.*

Proof. See [1]. □

Corollary 2.1.9. *A direct sum of a finite family of coherent sheaves is coherent.*

Proof. Apply the theorem to the exact sequence $0 \to \mathscr{F} \to \mathscr{F} \oplus \mathscr{G} \to \mathscr{G} \to 0$. □

Corollary 2.1.10. *Let $\phi : \mathscr{F} \to \mathscr{G}$ be a homomorphism from a coherent sheaf F to a coherent sheaf G . The kernel, the cokernel and the image of ϕ are also coherent sheaves.*

Proof. $\mathscr{F}$ is of finite type, so is the image of ϕ. Because image is a finite type subsheaf of $\mathscr{G}$ and $\mathscr{G}$ is coherent, image is coherent. For other two use the exact sequences

$$0 \to Ker(\phi) \to \mathscr{F} \to Im(\phi) \to 0$$
$$0 \to Im(\phi) \to \mathscr{G} \to Coker(\phi) \to 0.$$

□

Corollary 2.1.11. *If $\mathscr{F}$ and $\mathscr{G}$ are two coherent sheaves of $\mathcal{O}_X$-modules, $\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G}$ is coherent.*

Proof. By Lemma 2.1.4, locally we have the the exact sequence $\mathcal{O}_X^m|_V \to \mathcal{O}_X^n|_V \to \mathscr{F}|_V \to 0$. Thus we have the exact sequence $\mathcal{O}_X^m|_V \otimes_{\mathcal{O}_X} \mathscr{G}|_V \to \mathcal{O}_X^n|_V \otimes_{\mathcal{O}_X} \mathscr{G}|_V \to \mathscr{F}|_V \otimes_{\mathcal{O}_X} \mathscr{G}|_V \to 0$. But $\mathcal{O}_X^m|_V \otimes_{\mathcal{O}_X} \mathscr{G}|_V$ and $\mathcal{O}_X^n|_V \otimes_{\mathcal{O}_X} \mathscr{G}|_V$ are isomorphic to $\mathscr{G}^m|_V$ and $\mathscr{G}^n|_V$, since $\mathscr{G}$ is $\mathcal{O}_X$ module. Both of them are coherent so(coherence is a local property) $\mathscr{F}|_V \otimes_{\mathcal{O}_X} \mathscr{G}|_V$ is coherent for every open subset $U \subset X$. As a result $\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G}$ is coherent. □

2.1.3 Hom , Flasque, and Invertible sheaves

Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathscr{F}, \mathscr{G}$ two $\mathcal{O}_X$ modules.

Definition 2.1.12. *We can make a presheaf by $U \to Hom_{\mathcal{O}_X|_U}(\mathscr{F}|_U, \mathscr{G}|_U)$. This in fact a sheaf of abelian groups(why?????). We can make it an $\mathcal{O}_X$ module as follows. For $s \in \mathcal{O}_X(U)$ and $f \in Hom_{\mathcal{O}_X|_U}(\mathscr{F}|_U, \mathscr{G}|_U)$ define $sf \in Hom_{\mathcal{O}_X|_U}(\mathscr{F}|_U, \mathscr{G}|_U)$ by precomposing with multiplication by s on $\mathscr{F}_U$. We call the resulting sheaf the* ***Hom sheaf*** *$\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G})$.*

Note that the global sections of this sheaf is $Hom_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G})$.

For every $x \in X$ there is a canonical morphism $(\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G}))_x \to \mathscr{H}om_{\mathcal{O}_{X,x}}(\mathscr{F}_x, \mathscr{G}_x)$. This morphism is usually not an isomorshim but if $\mathscr{F}$ is coherent then it is an isomorphism.

Lemma 2.1.13. *Let $\mathscr{F}$ be a coherent sheaf, then the canonical map*

$$(\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F},\mathscr{G}))_x \to \mathscr{H}om_{\mathcal{O}_{X,x}}(\mathscr{F}_x,\mathscr{G}_x).$$

is an isomorphism.

A corrolary of the lemma is the following theorem.

Theorem 2.1.14. *If $\mathscr{F},\mathscr{G}$ are two coherent sheaves, then $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F},\mathscr{G})$ is a coherent sheaf.*

Proof. Coherence is a local notion. Since $\mathscr{F}$ is coherent, locally there is an exact sequence

$$\mathcal{O}_X^m \to \mathcal{O}_X^n \to \mathscr{F} \to 0.$$

Taking stalks and applying the *Hom* functor we get an exact sequence

$$0 \to \mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}_x,\mathscr{G}_x) \to \mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_{X,x}^n,\mathscr{G}_x) \to \mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_{X,x}^m,\mathscr{G}_x)$$

By the previous theorem, it is equivalent to

$$0 \to (\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F},\mathscr{G}))_x \to (\mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X^n,\mathscr{G}))_x \to (\mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X^m,\mathscr{G}))_x$$

Hence the exact sequence

$$0 \to \mathscr{H}om_{\mathcal{O}_X}(\mathscr{F},\mathscr{G}) \to \mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X^n,\mathscr{G}) \to \mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X^m,\mathscr{G})$$

We have $(Hom_{\mathcal{O}_X}(\mathcal{O}_X^n,\mathscr{G}) \cong \mathscr{G}^n$ and $\mathscr{H}om_{\mathcal{O}_X}(\mathcal{O}_X^m,\mathscr{G}) \cong \mathscr{G}^m$ which are coherent. Therefore $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F},\mathscr{G})$ is coherent. □

Definition 2.1.15. *A sheaf $\mathscr{F}$ on a topological space is* ***flasque*** *if every restriction map $\mathscr{F}(U) \to \mathscr{F}(V)$, $V \subset U$ is surjective.*

Theorem 2.1.16. *(i) if $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence, and $\mathscr{F}'$ is falsque, then for all open sets $U \subset X$ the sequence $0 \to \mathscr{F}'(U) \to \mathscr{F}(U) \to \mathscr{F}''(U) \to 0$ is exact.*

(ii) if $0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$ is an exact sequence, and if $\mathscr{F}'$ and $\mathscr{F}$ are flasque, then so is $\mathscr{F}''$.

(iii) if $X \to Y$ is a continuous map, and $\mathscr{F}$ is a flasque sheaf on X, then $f_*\mathscr{F}$ is flasque on Y.

Example 2.1.17. *On an irreducible topological space, any constant sheaf is flasque.*

The importance of Flasque sheaves is that they are injective objects in the category of $\mathcal{O}_X$-modules. We use them later to get a resolution and compute cohomology groups.

Finally we recal the notion of Invertible sheaves. Invertible sheaves are analogous notion of line bundles on a topological space.

Let $(X, \mathcal{O}_X)$ be a ringed space, and F an $\mathcal{O}_X$-module.

Definition 2.1.18. *F is called an* ***invertible sheaf*** *or* ***Locally free sheaf of rank 1*** *if for every point $x \in X$ there is an open neighborhood U such that $F|_U \cong \mathcal{O}_U$ as $\mathcal{O}_U$-modules. It is evident that invertible sheaves are quasi-coherent.*

Lemma 2.1.19. *If $\mathcal{O}_X$ is a coherent sheaf, so are all Invertible sheaves on X.*

Proof. Immediately follows from the lemma 2.1.7. □

In the next sections we will prove that the structure sheaves on schemes and complex spaces (by Oka theorem) are coherent. So:

Corollary 2.1.20. *(i) All invertible sheaves on schemes are coherent.*

(ii) All invertible sheaves on complex spaces are coherent.

2.2 Commutative Algebra

2.2.1 Topological ring, I-adic topology, completion

All of our rings are considered to be commutative with unit.

Definition 2.2.1. *A* ***topological ring*** *R is ring equipped with a topology on its underlying set, such addition and multiplication of ring structure are continuous functions.*

One way to define a topology on a commutative ring using its ring structure is by constructing a descending filtration of an ideal.

Every topological ring is an abelian topological group with respect to addition and hence a **uniform space.** Uniform spaces, invented by André Weil, are topological spaces with additional structure that is used to define uniform properties such as completeness, uniform continuity and uniform convergence. Roughly speaking they generalize the notion of a metric space.
More precisely we can define a Cauchy sequence in a (abelian)topological group with addition as the group operation: a sequence (r_i) is **Cauchy** if for every open neighborhood U of 0 (identity element of the group), there exist N such that for every $m, n \geq N$, $r_n - r_m \in U$. Now we say the topological group is complete if every Cauchy sequence converges.

Definition 2.2.2. *Completion of a ring is a completion of a topological ring to a complete topological ring. An important case of ring completion is adic completion with respect to a proper ideal.*

Definition 2.2.3. ***(I-adic topology)*** *Let I be a proper ideal of R. Powers of I^n are nested and define descending filtration on R, $R \supset I \supset I^2 \supset \cdots$. The bases of open neighborhoods of any $r \in R$ are given by $r + I^n$. Specially I^n are bases of open sets around 0. The resulting topology is called I-adic topology. The above notion of Cauchy sequence (r_i) becomes : for every n there exist N such that for every $m, n \geq N$, $r_n - r_m \in I^n$.*

Definition 2.2.4. *We say that R is* ***separated*** *in the I-adic topology if $\bigcap_n I^n = 0$.*

Adic completion (or formal completion) of an I-adic topology is $\widehat{R}_I = \varprojlim (R/I^n)$ in the category of commutative rings. In other words, an elements of $\widehat{R}_I$ is a sequences of elements $f_n \in R/I^n$ such that $f_n \equiv f_{n+1} \bmod I^n$, for all n.

Theorem 2.2.5. *The map $i : R \to \widehat{R}_I$ is a ring isomorphism if and only if R is complete in the I-adic topology.*

Note that in analaysis we define a completion of metric space by equivalence classes of Cauchy sequences. In fact this contructions is the same as I-adic completion.

Theorem 2.2.6. *Let $\mathcal{C}_{\mathcal{I}}(R)$ be the collection of Cauchy sequences on R with the I-adic topology and Let $\mathcal{C}_0(R)$ be the collection of Cauchy sequences in R that converge to 0. There is a canonical isomorphism:*

$$\frac{\mathcal{C}_{\mathcal{I}}(R)}{\mathcal{C}_0(R)} \cong \widehat{R}_I$$

Completion is a functorial operation: a continuous map $f : R \to S$ of topological rings gives rise to a map of their completions, $\widehat{f} : \widehat{R} \to \widehat{S}$.

Note that the case of $I = m$ where m is a maximal ideal is particularity important, for example in a local ring when we have a unique maximal ideal.

One motivation for defining formal completion as it is, is the formal power series rings.

Example 2.2.7. *The ring of p-adic integers $\mathbb{Z}_p$ is obtained by completing the ring $\mathbb{Z}$ of integers at the ideal (p).*

Example 2.2.8. *For R any ring and $R[x]$ the polynomial ring with coefficients in R, then the adic completion of $R[x]$ at the ideal (x) is the ring of formal power series R[[x]].*

Example 2.2.9. *Let R be a Noetherian ring and $I = (a_1, \dots, a_n)$ a finitely generated ideal. Then we have*

$$\widehat{R}_I = \frac{R[[x_1,\dots,x_n]]}{(x_1-a_1,\dots,x_n-a_n)}$$

Lemma 2.2.10. *Let R be a ring and $m \subset R$ a maximal ideal. Then $\widehat{R_m} = \widehat{R}$ where $\widehat{R}$ means completion along the maximal ideal m and $\widehat{R_m}$ means completion along the maximal ideal mR_m.*

Proof. By definition, $\widehat{R} = \varprojlim(R/m^n)$, and $\widehat{R_m} = \varprojlim(R_m/(mR_m)^n)$. We have $(R_m/(mR_m)^n) = (R/m^n)_m = R/m^n$, since R/m^n is a local ring. □

Theorem 2.2.11. *Let R be a ring and $m \subset R$ a maximal ideal. Then, the completion of R with respect to m is a local ring with unique maximal ideal $m\hat{R}$.*

Proof. We may represent an element of $x \in \hat{R}$ with $\sum_{i \geq 0} x_i$ with $x_i \in m^i$. We show the ideal $m\hat{R} = \{x | x_0 \in m\}$ which is the kernel of the projection map $\hat{R} \to R/m$ is the unique maximal ideal. It suffice to show that x with $x_0 \notin m$ is invertible. Suppose $x_0 = 1$ otherwise write $x = 1 + (x_0 y + x_1 y - 1) + \sum_{i \geq 2} x_i$ for $x_0 y - 1 \in m$. Now, the inverse of x is $x^{-1} = 1 + (x_1 + x_2 + \dots) + (x_1 + x_2 + \dots)^2 + \dots$. □

Completion of R-modules

Let $I \subset R$ an ideal.

Definition 2.2.12. *Like before, we can define the I-adic topology on an R-module M. A basis of open neighborhoods of a module M is given by the sets of the form $x + I^n M$ for $x \in M$.*

Definition 2.2.13. *The completion of an R-module M is the inverse limit of the quotients $\varprojlim(R/m^n)$. It converts any R-module into a complete topological module over $\widehat{R}_I$.*

Lemma 2.2.14. *Let R be Noetherian ring and $I \subset R$ an ideal. All completion is respect to the ideal I. then:*

(1) If $K \to N$ is an injective map of finite R-modules, then the map on completions $\hat{K} \to \hat{N}$ is injective.

(2) The completion functor is exact on the category of finite R-module.

(3) If M is a finite R-module, then $\hat{M} = M \otimes_R \hat{R}$.

2.2.2 Flat and faithfully flat morphisms

Now we present some preliminary results about modules which help us to prove GAGA theorems. All rings assumed to be commutative with unity.

Definition 2.2.15. *Let A be an R-module. We say A is **flat** if, for every exact sequence of R-modules*

$$0 \to E \to F \to G \to 0$$

the sequence

$$0 \to E \otimes_R A \to F \otimes_R A \to G \otimes_R A \to 0$$

is exact. In other words, the functor $M \mapsto M \otimes_R A$ is an exact functor.

Definition 2.2.16. *A homomorphism of rings $A \to B$ is called **flat** if one of the following equivalent conditions holds:*

(1) f makes B a flat A-module.

(2) $M \mapsto M \otimes_A B$ is an exact functor.

(3) for every injection of R-modules $N \to N'$ the map $M \otimes_R N \to M \otimes_R N'$ is injective.

(4) for every ideal $I \subset R$, the map $I \otimes_R M \to R \otimes_R M = M$ is injective.

Definition 2.2.17. *A morphism of rings $f : A \to B$ is called **faithfully flat** if it makes B a faithfully flat A-module. An R-module M is faithfully flat if one of the following equivalent conditions holds:*

(1) the sequence $E \to F \to G$ is exact if and only if $E \otimes_R A \to F \otimes_R A \to G \otimes_R A$ is exact.

(2) for every nonzero R-module N, the tensor product $M \otimes_R N$ is nonzero.

(3) M is flat and for every proper ideal $I \subset R$, $IM \neq M$.

(4) M is flat and for every maximal ideal $\mathfrak{m} \subset R$, $\mathfrak{m}M \neq M$.

Theorem 2.2.18. *A flat local homomorphism $f : A \to B$ of local rings is faithfully flat.*

Proof. Suppose m, n are the maximal ideals of A and B respectively. We have $f(m) \subset n$, so $f(m)B \subset n \neq B$. 2.2.17(4) finishes the theorem. □

Theorem 2.2.19. *Let I be an ideal of a Noetherian ring R. $\hat{R}$ means the completion with respect to I. Then the canonical map*

$$i : R \to \hat{R}$$

is flat.

Proof. Let $J \subset R$ be an arbitrary ideal. By lemma 2.2.14 $\hat{J} = J \otimes_R \hat{R} = \hat{R}($ R is noetherian). So the map $J \otimes_R \hat{R} \to R \otimes_R \hat{R} = \hat{R}$ is equivalent to the map $\hat{J} \to \hat{R}$. Since $J \to R$ is injective, by lemma 2.2.14, $\hat{J} \to \hat{R}$ is injective. Hence $R \to \hat{R}$ is flat by lemma 2.2.16. □

Theorem 2.2.20. *Let R be a Noetherian local ring, and $\hat{R}$ be its completion with respect to its unique maximal ideal. Then the canonical map $i : R \to \hat{R}$ is faithfully flat .*

Proof. Theorem 2.2.19 proves the flatness. Theorem 2.2.11 proves that $\hat{R}$ is a local ring and i is the local morphism. Theorem 2.2.18 gives the result. $\square$

Theorem 2.2.21. *Let $R_1 \xrightarrow{f} R_2 \xrightarrow{g} R_3$ be ring homomorphisms. If g and $g \circ f$ are faithfully flat, so is f.*

Theorem 2.2.22. *Let R and R' two local rings, and $\mathscr{I}$ an ideal of R. Let $\theta : R \to R'$ be a local morphism which can be extend to a map $\hat{\theta} : \hat{R} \to \hat{R}'$. If $\hat{\theta}$ is bijective, then the induced homomorphism $\hat{\theta} : \widehat{R/\mathscr{I}} \to \widehat{R'/\theta(\mathscr{I})R'}$ is also bijective.*

Proof. See [2]. $\square$

2.2.3 A distributive law for Hom and $\otimes$

The following lemma will be used in the proof GAGA theorems.

In certain situations and in some sense, tensor product is distributive with respective to Hom functor.

Lemma 2.2.23. *Let R be a Noetherian ring and S a flat R-algebra. Then for any finitely generated R-module A, any R-module B, the natural map*

$$\alpha : \mathrm{Hom}_R(A, B) \otimes_R S \to \mathrm{Hom}_S(A \otimes_R S, B \otimes_R S)$$

is a bijection.

Proof. (i) First suppose $A = R$, then $\mathrm{Hom}_R(R, B) = B$ and $\mathrm{Hom}_S(R \otimes_R S, B \otimes_R S) = B \otimes_R S$, hence the map α is the identity.
(ii) Next suppose that $A = \bigoplus_1^m R$ a finite rank free R module. Functors *Hom* and $\otimes$ commute with finite direct sum and the result follows from (i).
(iii) R is Noetherian and A a finitely generated R-module, hence it is finitely presented. Choose the presentation

$$F \to G \to A \to 0$$

where F, G are finite rank free R modules. Apply the functors $Hom_R(-, B)$ and $\otimes_R S$, then we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & Hom_R(A,B) \otimes_R S \arrow[r] \arrow[d] & Hom_R(G,B) \otimes_R S \arrow[r] \arrow[d] & Hom_R(F,B) \otimes_R S \arrow[d] \\
0 \arrow[r] & Hom_S(A \otimes_R S, B \otimes_R S) \arrow[r] & Hom_S(G \otimes_R S, B \otimes_R S) \arrow[r] & Hom_S(F \otimes_R S, B \otimes_R S)
\end{tikzcd}$$

The first row is exact since S is flat over R. The second row is exact since *hom* and $\otimes$ preserve exactness.

The second and third vertical arrows are isomorphisms by (ii), so is the first by the five lemma. $\square$

2.2.4 Nakayama's lemma

Nakayama's lemma is an important lemma in commutative algebra relating Jacobson radial of a ring and its finitely generated modules. We use this lemma in the third proof of GAGA theorem.

There are several form of the lemma:

Lemma 2.2.24. ***(Nakayama's lemma)*** *Let R be a commutative ring and M a finintly generated R-module. Then:*

(1) Let $I \subset R$ be an ideal, If $IM = M$, then there exists an $r \in R$ with $r \equiv 1$, such that $rM = 0$.

(2) If $J(R)$ is the Jacobson radical of R, and $J(R)M = M$, then $M = 0$.

(3) If N is a submodule of M, and $M = N + J(R)M$, then $M = N$.

Proof. See [10]. □

Corollary 2.2.25. *Let R be a local ring with $\mathfrak{m}$ the unique maximal ideal. If F is a submodule of a finitely generated R-module E, and we have $E = F + \mathfrak{m}E$, then $E = F$.*

2.3 Sheaf Cohomology Theory

Sheaf cohomology is applying the machinery of cohomology theory to sheaves on a topological space. Roughly speaking, sheaf cohomology describes the obstructions to solving a geometric problem globally when it can be solved locally. Sheaves and sheaf cohomology, were invented by Jean Leray at one of the prisoner-of-war camp in Austria. Soon, It became clear that sheaf cohomology was not only a new approach to cohomology in algebraic topology, but also a powerful method in complex analytic geometry used by Cartan, Serre and Oka. Armed with this deep knowledge of these techniques, Serre introduced sheaves and sheaf cohomology on algebraic varieties in his famous paper *FAC*. He used Cech cohomology to define sheaf cohomology. Grothendieck later give a more abstract definition of sheaf cohomology, now standard, using derived functors in the language of homological algebra. But this definition is rarely used directly to compute sheaf cohomology due to its abstractness, for this we often compute Cech cohomology which is an approximation to sheaf cohomology.

2.3.1 Sheaf cohomology

In this section we recall the notion of sheaf cohomology using derived functors and injective resolution. Our refrence for this section is [6].

First we have the following results.

Theorem 2.3.1. *Category* $\mathbf{Ab}$ *of abelian groups is an ablian category having enough injectives.*

Theorem 2.3.2. *Let* $(X, \mathcal{O}_X)$ *be a ringed space. Category* $\mathbf{Mod}(X)$ *of sheaves of* $\mathcal{O}_X$*modules is an ablian category having enough injectives.*

Corollary 2.3.3. *Category* $\mathbf{Ab}(X)$ *of sheaves of abelian groups is an ablian category having enough injectives by letting* $\mathcal{O}_X$ *the constant sheaf* $\mathbb{Z}$*.*

Note that we have a functor $\Gamma(X, \cdot) : \mathbf{Ab}(X) \to \mathbf{Ab}$ by $\mathscr{F} \to \mathscr{F}(X)$, namely the **functor of global sections**. This is a left exact functor. Now using derived functors we define the cohomology functors.

Definition 2.3.4. *Let the* ***cohomology functors*** $H^i(X, \cdot)$ *to be the right derived functors of* $\Gamma(X, \cdot)$*. For any sheaf* $\mathscr{F}$*, the groups* $H^i(X, \mathscr{F})$ *are the cohomology groups of* $\mathscr{F}$*. Even if* X *and* $\mathscr{F}$ *have some additional structure, e.g.,* X *a scheme* $\mathscr{F}$ *a coherent sheaf, we always take cohomology in this sense, regarding* $\mathscr{F}$ *simply as a sheaf of abelian groups on the underlying topological space* X*.*

If $(X, \mathcal{O}_X)$ is a ringed space, we can also define derived functors directly on the category of sheaves of $\mathcal{O}_X$-modules. These coincide with the functors $H^i(X, \cdot)$.

Theorem 2.3.5. *Let* $(X, \mathcal{O}_X)$ *be a ringed space. Then the derived functors of the functor* $\Gamma(X, \cdot)$ *from* $\mathbf{Mod}(X)$ *to* $\mathbf{Ab}$ *coincide with the cohomology functors* $H^i(X, \cdot)$*.*

Now the defintion of sheaf cohomology via derived functors are abstract. We need a more calculable way to compute the cohomology groups. This can be achieved by taking an Injective resolution of a sheaf.

Lemma 2.3.6. *If $(X, \mathcal{O}_x)$ is a ringed space. any injective $\mathcal{O}_x$-module is flasque.*

Theorem 2.3.7. *If $\mathscr{F}$ is flasque, then all higher cohomology groups vanish, i.e. $H^i(X, \mathscr{F}) = 0$ for all $i > 0$*

Proof. Embed $\mathscr{F}$ into an injective object $\mathscr{I}$, which is flasque by the above theorem, to get an exact sequence

$$0 \to \mathscr{F} \to \mathscr{I} \to \mathscr{K} \to 0$$

Since $\mathscr{F}, \mathscr{I}$ are flasque, by theorem 2.1.16 so is $\mathscr{K}$. By the same theorem

$$0 \to \mathscr{F}(X) \to \mathscr{I}(X) \to \mathscr{K}(\mathscr{X}) \to 0$$

is exact. $\mathscr{I}$ is injective, so $H^i(X, \mathscr{I}) = 0$ for all $i > 0$ by the properties of derived functors. By writing the long exact sequence of cohomology of the above sequence we see $H^i(X, \mathscr{F}) = H^{i-1}(X, \mathscr{K})$ and $H^1(X, \mathscr{F}) = 0$. The induction on i gives the result.

Corollary 2.3.8. *Injective sheaves are acyclic. Therefore, we can compute sheaf cohomology by taking an injective resolution of our ringed space.*

2.3.2 Cech cohomology

Cech cohomology is a well developed cohomology theory which can be explicitly computed by hand. In most of the cases and for all the objects we use in algebraic geometry like a variety or a scheme, this cohomology theory is the same as the sheaf cohomology. Therefore , we use this theory to compute cohomology groups in the proof of theorems. Our reference for this section is [1].

Let X be a topological space and $\mathscr{F}$ be a sheaf of abelian groups on X. First we assign a complex to each open cover of X.

Let $\mathfrak{U} = \{U_i\}_{i \in I}$ be an open cover of X. For simplicity, Assume the set I is equipped with a total ordering. For each finite subset J of I, we define $U_J = \cap_{i \in J} U_i$, with $U_\emptyset = X$. Let $\check{C}^j(\mathfrak{U}, \mathscr{F})$ be the direct product of $\Gamma\left(\mathscr{F}, U_J\right)$ over all $(j+1)$ -element subsets J of I. An element in $\check{C}^j(\mathfrak{U}, \mathscr{F})$ is called a j-cochain, it can be viewed as a function f assigning to every $s = (i_0, \dots, i_j)$ of $j+1$ elements of I a section $f_s = f_{i_0 \dots i_j}$ of $\mathscr{F}$ over $U_{i_0 \dots i_j}$.

The **differential** $d^j : \check{C}^j(\mathfrak{U}, \mathcal{F}) \to \check{C}^{j+1}(\mathfrak{U}, \mathcal{F})$ is defined as follows: for $f \in \check{C}^j(\mathfrak{U}, \mathscr{F})$ we have

$$(d^j f)_{i_0 \dots i_{j+1}} = \sum_{p=0}^{p=j+1} (-1)^p \rho_p \left(f_{i_0 \dots \hat{i}_p \dots i_{j+1}} \right)$$

where we have the restriction maps

$$\rho_p : \Gamma\left(U_{i_0 \dots \hat{i}_p \dots i_{j+1}}, \mathscr{F}\right) \to \Gamma\left(U_{i_0 \dots i_{j+1}}, \mathscr{F}\right)$$

Because of $(-1)^p$ signs it's easy to see $d \circ d = 0$. Therefore $C(\mathfrak{U}, \mathscr{F})$ is equipped with a coboundary operator d which makes it a complex.

Definition 2.3.9. ***(the Cech complex associated to*** $\mathfrak{U}$***).*** *Denoted* $C(\mathfrak{U},\mathscr{F})$ *the family* $\check{C}^j(\mathfrak{U},\mathscr{F}), j = 0,1,\ldots,$ *, we find that* $C(\mathfrak{U},\mathscr{F})$ *is equipped with a coboundary operator making it a complex.*

$$\check{C}^0(\mathfrak{U},\mathscr{F}) \to \check{C}^1(\mathfrak{U},\mathscr{F}) \to \check{C}^2(\mathfrak{U},\mathscr{F}) \to \ldots$$

The q -th cohomology group of the complex $C(\mathfrak{U},\mathscr{F})$ will be denoted by $\check{H}^q(\mathfrak{U},\mathscr{F})$. We have:

Theorem 2.3.10. $\check{H}^0(\mathfrak{U},\mathscr{F}) = \Gamma(X,\mathscr{F})$.

Proof. By definition $\check{H}^0(\mathfrak{U},\mathscr{F})$ contains 0-chains f which $d(f) = 0$. A 0-chain is a system of $\{f_i\}_{i\in I}$ where $f_i \in \Gamma(U_i, X)$. $d(f) = 0$ means $f_i - f_j = 0$ over $U_i \cap U_j$ for every i, j. By the gluing property of sheaves there exist a global section $f' \in \Gamma(X,\mathscr{F}), f'|_{U_i} = f_i$.□

Now we try to make a direct system of open covers by the notion of finer cover. more precisely:

A covering $\mathfrak{U} = \{U_i\}_{i\in I}$ is said to be **finer** than the covering $\mathfrak{V} = \left\{V_j\right\}_{j\in J}$ if there exists a mapping $\tau : I \to J$ such that $U_i \subset V_{\tau i}$ for all $i \in I$. If $f \in C^q(\mathfrak{V},\mathscr{F})$, put

$$(\tau f)_{i_0,\ldots,i_q} = \rho_U^V\left(f_{\tau i_0 \ldots \tau i_q}\right)$$

ρ_U^V denoting the restriction homomorphism defined by the inclusion of $U_{i_0\ldots i_\varphi}$ in $V_{\tau i_0\ldots\tau i_q}$. The mapping $f \mapsto \tau f$ is a homomorphism from $C^q(\mathfrak{V},\mathscr{F})$ to $C^q(\mathfrak{U},,\mathscr{F})$, defined for all $q \geq 0$ and commuting with d, thus it defines homomorphisms

$$\tau^* : H^q(\mathfrak{V},\mathscr{F}) \to H^q(\mathfrak{U},\mathscr{F})$$

It can be shown the homomorphisms $\tau^* : H^q(\mathfrak{V},\mathscr{F}) \to H^q(\mathfrak{U},\mathscr{F})$ depend only on $\mathbb{U}$ and $\mathfrak{B}$ and not on the chosen mapping τ.

As a result, if $\mathfrak{U}$ is finer than $\mathfrak{V}$, there exists for every integer $q \geq 0$ a canonical homomorphism from $H^q(\mathfrak{V},\mathscr{F})$ to $H^q(\mathfrak{U},\mathscr{F})$. Using refinements, the coverings of X form a direct system. Because we are working with abelian groups, which admit colimits, we can form the direct limit

$$\check{H}^q(X,\mathscr{F}) = \underset{\overrightarrow{\mathfrak{U}}}{lim}\check{H}^q(\mathfrak{U},\mathscr{F}).$$

Definition 2.3.11. *We call* $\check{H}^q(X,\mathscr{F})$ *the cech cohomology groups.*

Corollary 2.3.12. $\check{H}^0(X,\mathscr{F}) = \Gamma(X,\mathscr{F})$.

The alternating Cech complex

We don't need to have the whole $\check{C}^j(\mathfrak{U},\mathscr{F})$ to compute cohomology, we only need a subcomplex of it, the altenrating Cech complex.

Definition 2.3.13. *For an open cover of* $X = \bigcup_{i\in I}(U_i)$ *define* $\check{C}^j_{alt}(\mathfrak{U},\mathscr{F})$ *the alternating Cech complex associated to* $\mathscr{F}$ *as:*

$$\check{C}^j_{alt}(\mathfrak{U},\mathscr{F}) = \{s \in \check{C}^j(\mathfrak{U},\mathscr{F}) | s_{i_0 i_1,\ldots,i_p} = 0 \text{ if } i_l = i_r, r \neq l, and$$
$$s_{i_0,\ldots,i_m,\ldots,i_n,\ldots,i_p} = -s_{i_0,\ldots,i_n,\ldots,i_n,\ldots,i_p} \text{ for all indices. }\}$$

Theorem 2.3.14. *The inclusion map $\check{C}^{\bullet}_{alt}(\mathfrak{U},\mathscr{F}) \to \check{C}^{\bullet}(\mathfrak{U},\mathscr{F})$ is a homotopy equivalence.*

Let $\check{H}^{q}_{alt}(X,\mathscr{F})$ be the cohomology groups of the complex $\check{C}^{\bullet}_{alt}(\mathfrak{U},\mathscr{F})$.

Corollary 2.3.15. *There exist an isomorphism $\check{H}^{q}_{alt}(X,\mathscr{F}) = \check{H}^{q}(X,\mathscr{F})$ for all $q \geq 0$*

Corollary 2.3.16. *If $X = \bigcup_{i\in I}(U_i)$ is the open cover consists of a finite number of open sets, and $|I| = t$, then $\check{H}^{q}(X,\mathscr{F}) = 0$ for all $q > t$.*

Proof. We have $s_{i_0 \dots i_q} = 0$ for $q > t$ and distinct indices $i_0, \dots i_q$. Hence $\check{C}^{q}_{alt}(\mathfrak{U},\mathscr{F}) = 0$ which gives $\check{H}^{q}(X,\mathscr{F}) = 0$. □

2.3.3 Comparison of Cech and sheaf cohomology

Until now we have define two cohomology, one using derived functors namely sheaf cohomology and the other with covering namely Cech cohomology. In general these two cohomology are not equal but under certain circumstances, we can show that Cech cohomology computes sheaf cohomology. There are two major theorems relating these two cohomology, the first one has conditions on Cech cohomology groups, the second one has conditions on Sheaf cohomology groups. Our reference is [12], the sheaf cohomology notes.

Theorem 2.3.17. ***(Cartan's theorem)*** *Let X be a topological space $\mathscr{F}$ be sheaf on it. Let $\mathcal{U}$ a basis of open sets, stable under finite intersections. If **Cech cohomology** groups of $\mathscr{F}$ vanish, i.e,$\check{H}^{i}(U,\mathscr{F}) = 0$ for all $U \in \mathcal{U}$ and $i > 0$, Then the Cech cohomology of $\mathscr{F}$ on X coincides with sheaf cohomology.*

We say the cover $\mathfrak{U}$ is good for $\mathscr{F}$ if for each finite J, $\mathscr{F}|_{U_J}$ is **acyclic**, it means that **sheaf cohomology** groups $H^{p}(\cap_{i\in J}U_i,\mathscr{F}) = 0$ for all $p > 0$.

Theorem 2.3.18. ***(Leray)***
If $\mathfrak{U}$ is good for $\mathcal{F}$, then the morphisms $\check{H}(\mathfrak{U},\mathscr{F}) \to H(X,\mathscr{F})$ are isomorphisms. That is, the Cech complex $\check{C}(\mathfrak{U},\mathscr{F})$ computes the sheaf cohomology of $\mathscr{F}$.

2.4 Algebraic Geometry

Two central theorems in the proof of GAGA theorems are Serre's finiteness and Vanishing theorems. Our goal in this section is to prove those theorems, but before that, we need to know a quite much about the sheaf cohomology for varieties and schemes. We also need to know about the Twisting sheaves.

Almost all of the results we want to prove hold for both the schemes and varieties. The proofs are almost identical in both cases. Therefore we only deal with one of them. We choose to work with the general framework, which is the scheme.

2.4.1 Schemes, Proj, and morphisms of schemes

We briefly recall the notion of an scheme. First we define the local model, namely the affine schemes.

Definition 2.4.1. ***(Affine scheme)***
Let R be a ring. We want to make it to a ringed space.

As a topological space: *Define Spec (R) to be the set of prime ideals $\mathfrak{p}$ of R. For $S \subset R$ let $V(S) = \{\mathfrak{p} \in Spec\ R | S \subset \mathfrak{p}\}$. We declare the sets $V(S)$ the close sets of the topology which we call the Zarisky topology. Note that the set $(D_f)_{f \in R}$ for $(D_f) = \{\mathfrak{p} \in Spec\ R | f \notin \mathfrak{p}\}$ are a base of the topology.*

Structure sheaf: *Define $\mathcal{O}(D_f) = R_f$. The restriction maps defined in the obvious way. it defines a sheaf on D_f. It can be seen that the sheaves on $(D_f)_{f \in R}$ glue together and give a sheaf on $SpecR$ called the structure sheaf $\mathcal{O}_{SpecR}$. Note that we have $\mathcal{O}_{\mathfrak{p},SpecR} = R_{\mathfrak{p}}$.*
The resulting locally ringed space is called an affine scheme

Definition 2.4.2. ***(Scheme)*** *An scheme is a locally ringed space $(X, \mathcal{O}_X)$ which every point $x \in X$ has an open neihgbourhood U such that $(U, \mathcal{O}_X|_U)$ is an affine scheme. The morphism of schemes is a morphism of locally ringed space.*

Now we want to show the correspondence between ideal sheaves and closed subschemes of a scheme.

Definition 2.4.3. *Let $Y \subset X$ be a closed subspace and $i : Y \to X$ be the inclusion. Define $\mathscr{I}_Y$ the **ideal sheaf of** Y to be the kernel of $i^\sharp : \mathcal{O}_X \to \mathcal{O}_Y = i_*\mathcal{O}_Y$. So we have the exact sequence*

$$0 \to \mathscr{I}_Y \to \mathcal{O}_X \to \mathcal{O}_Y \to 0.$$

Theorem 2.4.4. *For any closed subscheme Y of X, $\mathscr{I}_Y$ is a quasi-coherent sheaf of ideals. Conversely, any quasi-coherent sheaf of ideals on X is of the form of $\mathscr{I}_Y$ for some uniquely determined closed subscheme $Y \subset X$*

Proj of a graded ring

The GAGA theorems are only valid for the projective schemes and varieties, although we will see there are far reaching generalizations of them. Therefore, our main objects of study are the projective schemes or more generally the projecive spaces associated to graded rings.

Let $S = \bigoplus_{d\geq 0} S_d$ be a graded ring, and $S_+ \subset S$ be positive degree elements. We are going to construct a locally ringed space called Proj(S) as follows.

As a topological space: Define Proj(S) to be the set of homogeneous prime ideals p of S such that $S_+ \notin p$. The set Proj(S) is a subset of Spec(S) and we endow it with the induced topology.

Let $f \in S_+$. Define

$$D_+(f) = \{p \in Proj(S) | f \notin p\} = Spec((S)_f)_0.$$

Then $(D_+(f))_{f\in S_+}$ is an open basis of Proj(S), which we call standard open sets. Note that $D_+(f) \cap D_+(g) = D_+(fg)$.

Structure sheaf: Equip $(D_+(f))$ with the structure sheaf of $Spec((S)_f)_0$. Then structure sheaves $Spec((S)_f)_0$ and $Spec((S)_g)_0$ are isomorphism on the intersection $Spec((S)_{fg})_0$, and these gluings behave well on triple overlap. We then glue the various $Spec((S)_f)_0$ $f \in S_+$ altogether, using these pairwise gluings.

Definition 2.4.5. *Let R be a ring. We define projective space $\mathbb{P}^n_R = ProjR[x_0, \ldots, x_n]$*

Separated, finite type, proper, and projective morphisms

Definition 2.4.6. *A morphism $f : X \to Y$ of schemes is **of finite type** if there exist a cover of Y by affine open sets $V_i = SpecB_i$ such that $f^{-1}(V_i)$ can be covered by a finite number of open affine subsets $U_{ij} = SpecA_{ij}$ with each A_{ij} a finitely generated B_i-algebra.*

Definition 2.4.7. *A morphism $f : X \to Y$ of schemes is **separated** if the diagonal morpohism $\triangle : X \to X \times_Y X$ is a closed immersion. This is equivalent to that image of diagonal moprhism is a closed subset of $X \times_Y X$.*

Definition 2.4.8. *A morphism $f : X \to Y$ of schemes is **universally closed** if it is closed, and for any morphism $Y' \to Y$ the fiber product projection map π*

$$\begin{array}{ccc} X \times_Y Y' & \longrightarrow & X \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle f} \\ Y' & \longrightarrow & Y \end{array}$$

is closed.

Definition 2.4.9. *A morphism $f : X \to Y$ of schemes is **proper** if it is separated, of finite type and universally closed. We say X is proper over Y.*

Example 2.4.10. *(i) projective space $\mathbb{P}^n_R$ is proper over R.*
(ii) closed immersions are proper.
(iii) proper morphisms are stable under composition and base extension.

The importance of proper morphisms for us is that the proper morphisms between locally Noetherian schemes preserve coherent sheaves. We use this when we want to prove analogous GAGA theorem for formal schemes.

Remark 2.4.11. *Proper morphism between schemes is the analogue of proper morphisms between topological spaces, where the inverse image of a compact set is compact. In fact a morphism $f : X \to Y$ between finite type schemes over $\mathbb{C}$ is proper if and only if the corresponding analytic map is a topological proper morphism.*

Note that for any ring R we define projective space $\mathbb{P}^n_R = ProjR[x_0, \ldots, x_n]$. If $A \to B$ is a homomorphism of rings , then $\mathbb{P}^n_B \cong \mathbb{P}^n_A \times_{SpecA} SpecB$. By generalizing this definition we get:

Definition 2.4.12. *Let Y be a scheme. The projective space over Y is $\mathbb{P}^n_Y = \mathbb{P}^n_Z \times_{SpecZ} Y$.*

Definition 2.4.13. *A morphism $f : X \to Y$ of schemes is **projective** if it factors through $X \xrightarrow{i} \mathbb{P}^n_Y \xrightarrow{\pi} Y$ for some n, where i is a closed immersion and π is the projection map.*

Definition 2.4.14. *A morphism $f : X \to Y$ of schemes is **quasi-projective** if it factors through $X \xrightarrow{i} X' \xrightarrow{p} Y$, where i is an open immersion and p is a projective morphism.*

Lemma 2.4.15. *(1) A projective morphism of Noetherian schemes is proper.*

(2) A quasi-projective morphism of Noetherian schemes is of finite type and separated.

2.4.2 (quasi)coherent sheaves on schemes

We define (quasi)coherent sheaves on a ringed space in the section 2.1. In the case of schemes, definitions are equivalent to some easier ones.

Let R be a ring and M be an R-module. We define the sheaf associated to M on Spec R, namely $\tilde{M}$.

Definition 2.4.16. ***(Sheaf associated to a module)*** *$\tilde{M}$ is constructed as follows. For each prime ideal $p \subset R$, let M_p the localization of M at p. Define $\tilde{M}(U)$ for open set $U \subset$Spec R as the set of functions $f : U \to \coprod_{p \in U} M_p$ with $f(p) \in M_p$, such that for each $p \in U$, there is a neighborhood V of p, $m \in M$, and $r \in R$ such that for each $q \in V$ and $r \notin q$, $s(q) = \frac{m}{r}$ in M_q. We can make it to a sheaf with canonical restrictions maps.*

It can be easily seen that $\tilde{M}$ is an $\mathcal{O}_X$ module.

Lemma 2.4.17. *Th map $M \mapsto \tilde{M}$ is an exact and fully faithful functor from the category of R-modules to the category of $\mathcal{O}_X$ modules.*

Lemma 2.4.18. *Let $(X, \mathcal{O}_X)$ be a scheme. A sheaf of $\mathcal{O}_X$ module $\mathscr{F}$ is **quasi coherent** if and only if over each open affine subscheme $U =$SpecR we have $\mathscr{F}|_U \cong \tilde{M}$ for $M = \Gamma(U, \mathcal{F})$.*

The notion of a **coherent** sheaf is well behaved only in locally noetherian schemes.

Theorem 2.4.19. *Let $(X, \mathcal{O}_X)$ be a locally Noetherian schemes. A sheaf of $\mathcal{O}_X$ module $\mathscr{F}$ is coherent if and only if $\mathscr{F}$ is quasi-coherent and the modules M defined above can be taken to be finitely generated.*

It is worth mentioning that on an affine scheme Spec R, there is an equivalence of categories from category of R-modules to quasi-coherent sheaves by the functor $M \mapsto \tilde{M}$. The inverse of this functor is the global section functor. If R is Noetherian the same functor gives an equivalence of categories between finitely generated R-module and coherent sheaves.

Example 2.4.20. *By above lemma it is obvious that on any scheme $(X, \mathcal{O}_X)$, the structure sheaf $\mathcal{O}_X$ is quasi coherent, and coherent in the case of locally Noetherian scheme.*

Lemma 2.4.21. *Let $f : X \to Y$ be a morphism of schemes. If $\mathscr{G}$ is a quasi-coherent sheaf of $\mathcal{O}_Y$-modules, $f^*\mathscr{G}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules. If X, Y are noetherian, the same is true for coherent sheaves.*

(quasi)coherent sheaf on Proj of a graded ring

Let S be a graded ring and let M be a graded S-module. Define the sheaf associated to M on Proj S, $\tilde{M}$, as follows:

Definition 2.4.22. *For each $p \in \operatorname{Proj} S$, let $M_{(p)}$ be the group of elements of degree 0 in the localization $T^{-1}M$, where T is the multiplicative system of homogeneous elements of S not in p. For any open subset $U \subseteq$ Proj S we define $\tilde{M}(U)$ to be the set of functions $s : U \to \coprod_{p \in U} M_{(p)}$ such that for every $p \in U$, there is a neighborhood V of p in U, and homogeneous elements $m \in M$ and $f \in S$ of the same degree, such that for every $q \in V$, we have $f \notin \mathrm{q}$, and $s(q) = m/f$ in $M_{(q)}$. Then make $\tilde{M}$ into a sheaf with the canonical restriction maps.*

Theorem 2.4.23. *If S is a graded ring with M a graded S-module and $X =$ Proj S, then $\tilde{M}$ is a quasi-coherent $\mathcal{O}_x$-module, If S is Noetherian and M is finitely generated, then $\tilde{M}$ is coherent.*

The inverse of the previous theorem holds when S is finitely generated algebra by S_1:

Theorem 2.4.24. *Suppose S is finitely generated by S_1 as an S_0-algebra and $\mathscr{F}$ a quasi-coherent sheaf on Proj S. Then there is a canonical choice of M such that $\tilde{M} \cong \mathscr{F}$.*

2.4.3 Cohomology of schemes

Lemma 2.4.25. *If X is a separated scheme(or variety) , then for $U, V \subset X$ open affine, the intersection $U \cap V$ is affine.*

The following theorem is a fundamental theorem of affine schemes, which shows that the analogue of contractible open subsets in the topological case are affine schemes, on which quasicoherent sheaves are acyclic.

Theorem 2.4.26. *If X is an affine scheme(or affine variety), and $\mathscr{F}$ a quasi-coherent $\mathcal{O}_X$-module. Then $\mathscr{F}$ is acyclic, i.e., $H^p(X, \mathscr{F}) = 0$ for all $p > 0$.*

Sketch of the proof. Let $X = SpecR$. We only prove the case of R being Noetherian. For an arbitrary affine scheme see [5]. Let $M = \Gamma(x, \mathscr{F})$, since $\mathscr{F}$ is quasicoherent we know that $\mathscr{F} = \tilde{M}$. Take an injective resolution. $0 \to M \to I^\bullet$ in the category of R-modules. Consider the correspoding exact sequence of sheaves $0 \to \tilde{M} \to \tilde{I}^\bullet$. The hard part is the following lemma:

Lemma 2.4.27. *If I is an ijective module over a Noetherian ring R, then the sheaf $\tilde{I}$ on $SpecR$ is flasque.*

Suppose the lemma holds. Then $\tilde{I}^i$ is flasque for all i, hence we can use the sequence $0 \to \tilde{M} \to \tilde{I}^\bullet$ to compute cohomology. It is clear that by taking global sections of this sequence we get back the injective resolution $0 \to M \to I^\bullet$. Therefore, $H^i(X, \mathscr{F}) = 0$ for $i > 0$.

The lemma is nontrivial and its proof goes by Noetherian induction and usage of some commutative algebra results. For proof see [6]. Note that the Noetherian hypothesis is cruciall here. If we drop this assumption, then it is not true that the $\tilde{I}^i$ is flasque. So we cant use the mentioned sequence to compute cohomology. Another way to prove the general theorem is to compute Cech cohomology instead of sheaf cohomology, that is how it is done in EGA.

□

Lemma 2.4.28. *Let X be a scheme. Let $\mathcal{U} = (U_i)_{i \in I}$ be an open covering such that $U_{i_0} \cap \ldots \cap U_{i_p}$ is open affine for all $p \geq 0$ and all $i_0, \ldots, i_p \in I$. Then Cech cohomology with respect to this cover computes sheaf cohomology, i.e: for any quasi-coherent sheaf $\mathscr{F}$ we have*

$$\check{H}^j(\mathcal{U}, \mathscr{F}) = H^j(X, \mathscr{F})$$

for all $j \geq 0$.

Proof. This is a direct consequence of Leray theorem 2.3.18 and the vanishing theorem of affine sets 2.4.26. □

Lemma 2.4.29. *Let X be a quais-compact, separated scheme(or vareite). If **t** is the minimal number of affine open sets needed to cover X, then $H^n(X, \mathscr{F}) = 0$ for all $n \geq t$ and all quasi-coherent sheaves.*

Proof. Suppose $\mathcal{U} = (U_i)$ is a cover of X by t open affine sets. Since X is separated, all finite intersections of (U_i) are also affine. By lemma 2.4.28 $\check{H}^j(\mathcal{U}, \mathscr{F}) = H^j(X, \mathscr{F})$. Corollary 2.3.16 finishes the proof. □

Sheaf cohomology commutes with direct-limits of sheaves

In general Sheaf cohomology does not commute with direct-limit, but in the case of noetherian topological space, it does hold.

Theorem 2.4.30. *Let X be a Noetherian topological space, and let $(\mathscr{F}_\alpha)$ be a direct system of abelian sheaves. Then there are natural isomorphisms, for each $i \geqslant 0$*

$$\varinjlim_{n} H^i(X, \mathscr{F}_\alpha) \to H^i\left(X, \varinjlim_{n} \mathscr{F}_\alpha\right)$$

Corollary 2.4.31. *In the case of Noetherian schemes, sheaf cohomology commutes with direct-limit of sheaves.*

Remark 2.4.32. *Sheaf cohomology does not commute with limit(inverse-limit), we later give a sufficient condition for this using Mittag-Leffler condition.*

2.4.4 Twisting sheaf of Serre

Twisting sheaves were introduced by Serre in his famous paper FAC.

We first motivate the definition of twisting sheaf by explaining them on the projective space as a variety. Let n be an integer.

Consider the projective space $(P^n_K, \mathcal{O})$ with its sheaf of regular functions and $x_1, \ldots, x_n$ as its coordinates. P^n_K can be covered by affine varieties $U_i = \{x_i \neq 0\}$. Now restrict the sheaf to U_i to obtain $\mathcal{O}_i = \mathcal{O}|_{U_i}$. Now the idea is to glue $\mathcal{O}_i$ together differently to obtain a new sheaf. More precisely, glue $\mathcal{O}_i$ to $\mathcal{O}_j$ by the isomorphism $s \mapsto s \times t_j^n / t_i^n$ over $U_i \cap U_j$. These isomorphism are compatible and gives rise to a sheaf, namely, $\mathcal{O}(n)$.

In addition, sections of $\mathcal{O}(n)$ over an open set V can be seen as regular functions of degree n; a section over V consists of data (f_i) of zero degree regular functions over $V \cap U_i$, such that $f_i = t_j^n / t_i^n \times f_j$, setting $s_i = t_i^n f_i$, it can be seen s_i are compatible and glue together to get a regular functions of degree n over V. In particular, by letting $f_i = x_k^n / x_i^n$ for $i \neq k$ and $f_k = 1$ we see that x_k^n is a global section of $\mathcal{O}(n)$ for $n > 0$. However for $n < 0$, $\mathcal{O}(n)$ has no global section.

Therefore, twisting a sheaf shifts the grading of the space of functions. It is a well known fact that global sections over P^n_K are constant functions, despite one usually expects coordinate functions x_i play a role. x_i are not global sections because they are not defined on the whole space. In contrast, x_i are global section for $\mathcal{O}(1)$. In general, global sections of $\mathcal{O}(n)$ are precisely the vector space of homogeneous polynomial of degree n. As a result, we can recover the graded ring $K[x_0, \ldots, x_n]$ by twisting sheaves $\mathcal{O}(n)$.

Where does the name **twisting** comes from? This is mainly because invertible sheaves in algebraic geometry correspond to topological line bundles. Now if $K = \mathbb{C}$, then $\mathcal{O}(1)$ would correspond to **Mobius strip**! Roughly speaking, $\mathcal{O}(n)$ corresponds to a bundle with n twist.

We can define twisted sheaves more precisely on Proj of a graded ring.

Definition 2.4.33. *Let S be a graded ring and an integer $k \geq 0$. Let M be a graded S module. Define the shifted module $M(k)$ as, $M(k)_i = M_{k+i}$. Define the* ***twisted sheaf*** *$\mathcal{O}_X(k) = \widetilde{S(k)}$. It is a quasi coherent sheaf by the previous section. For any quasi-coherent sheaf $\mathscr{F}$ of $\mathcal{O}_X$-modules, define $\mathscr{F}(k) = \mathscr{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(k)$.*

Lemma 2.4.34. *Suppose that S is generated by S_1 as an S_0-algebra. Then* *(i) The sheaves $\mathcal{O}_X(k)$ are locally free of rank 1(hence an invertible sheaf) with the converse $\mathcal{O}(-k)$. In other words, $\mathcal{O}_X(k) \otimes_{\mathcal{O}_X} \mathcal{O}_X(-k) \cong \mathcal{O}_X$.*

(ii) For a graded S-module M, $\tilde{M}(k) \cong \widetilde{(M(k))}$. So $\mathcal{O}_X(k) \otimes_{\mathcal{O}_X} \mathcal{O}_X(k') \cong \mathcal{O}_X(k+k')$.

Lemma 2.4.35. *Let $X = P^n_R$ for a Noetherian ring R. If $\mathscr{F}$ is a coherent $\mathcal{O}_X$-module, so is $\mathscr{F}(k)$.*

Proof. R is Noetherian so is $M = R[x_0, \ldots, x_n]$, hence $\mathcal{O}_X(k) = \widetilde{M(k)}$ is coherent by theorem 2.4.19. Since $\mathscr{F}(k) = \mathscr{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(k)$ and $\mathscr{F}$ is coherent, $\mathscr{F}(k)$ is coherent. □

Note that each homogeneous element $s \in S$ of degree d determines a global section $d \in \Gamma(X, \mathcal{O}_X(d))$. This is because degree zero of each graded ring determines a global section, now, zero degree elements of $S(d)$ is d-degree elements of S.

From now on we focus our attention to Projective spaces $X =$Proj S, where $S = R[x_0, \ldots, x_n]$ for a ring R. We can see that $x_0, \ldots, x_n$ are global sections on $\mathcal{O}_X(1)$, or more generally each polynomial of degree d gives rise to a global section on $\mathcal{O}_X(d)$. It turns out that for the case $S = R[x_0, \ldots, x_n]$, these are the only global sections. More precisely:

Theorem 2.4.36. *Let $n \geq 1$. For $S = R[x_0, \ldots, x_n]$ with the usual grading, we have*

$$S = \oplus_0^\infty \Gamma(X, \mathcal{O}_X(k))$$

Lemma 2.4.37. *Tensoring by an invertible object is an equivalence of categories, hence, it preserves exact sequences.*

Corollary 2.4.38. *Twisting on $\mathbf{P}^n_R$ is an exact functor, in other words, if*

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$$

is an exact sequence of sheaves on $\mathbf{P}^n_R$, the twisted sequence

$$0 \to \mathscr{F}'(n) \to \mathscr{F}(n) \to \mathscr{F}''(n) \to 0$$

is also exact.

Proof. $\mathcal{O}_X(k)$ are invertible sheaves. □

Theorem 2.4.39. *let $X = \mathbf{P}^n_R$ the projective scheme or $X = \mathbf{P}^n$ as the projective vareiety. Let $f \in \mathcal{O}_X(k)$ be a homogeneous degree k polynomial and define E to be the hypersurface $f = 0$. Then there exist an exact sequence of $\mathcal{O}_X$*

$$0 \to \mathcal{O}_X(-k) \xrightarrow{\times f} \mathcal{O}_X \to (\mathcal{O}_X)_E \to 0.$$

Proof. Let $S = R[x_0, \ldots, x_n]$. we have an exact sequence of graded modules

$$0 \to S(-k) \xrightarrow{\times f} S \to S/(f) \to 0$$

The functor $(\tilde{-})$ which associated a sheaf to graded module is exact. Applying it to the above sequence gives the result.

□

2.4.5 Serre's finiteness and vanishing theorems

In this section let $X = P_K^n$ as a projective variety for a field K, or $X = P_R^n$ as a projective scheme for a Noetherian ring R.

Theorem 2.4.40. *Let R be a ring. For $n \geq 0$ we have:*

$$H^q\left(\mathbf{P}_R^n, \mathcal{O}_{\mathbf{P}_R^n}(d)\right) = \begin{cases} (R[x_0, \ldots, x_n])_d & \text{if} \quad q = 0 \\ 0 & \text{if} \quad q \neq 0, n \\ \left(\frac{1}{x_0 \ldots x_n} R\left[\frac{1}{x_0}, \ldots, \frac{1}{x_n}\right]\right)_d & \text{if} \quad q = n \end{cases}$$

as R-modules.

Sketch of the proof. We compute the cohomology using Cech complex with respect to the standard affine covering $\mathcal{U}$, $\mathbf{P}_R^n = \bigcup_0^n D_+(x_i)$. Use the usual ordering on $0, \ldots, n$. We have $D_+(x_{i_0}) \cap \ldots \cap D_+(x_{i_p}) = D_+(x_{i_0} \ldots x_{i_p})$. Therefore

$$\check{C}^p\left(\mathcal{U}, \mathcal{O}_{\mathbf{P}_R}(d)\right) = \bigoplus_{i_0 < \ldots < i_p} \left(R\left[x_0, \ldots, x_n, \frac{1}{x_{i_0} \ldots x_{i_p}}\right]\right)_d$$

Now we should use the the definition of differential

$$d(s)_{i_0 \ldots i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j s_{i_0 \ldots \hat{i}_j \ldots i_{p+1}}$$

and compute each cohomology based on the value of q. This is not hard but a boring computation. □

Corollary 2.4.41. *The cohomology groups $H^i(\mathbf{P}_R^n, \bigoplus_{1 \leq j \leq t} \mathcal{O}_{\mathbf{P}_R^n}(d_j))$ are finitely-generated R-modules.*

Proof. This is because projective space is Noetherian and cohomology functors preserve colomits, therefore, $H^i(\mathbf{P}_R^n, \bigoplus_{1 \leq j \leq t} \mathcal{O}_{\mathbf{P}_R^n}(d_j)) = \bigoplus_{1 \leq j \leq t} H^i(\mathbf{P}_R^n, \mathcal{O}_{\mathbf{P}_R^n}(d_j))$. □

Theorem 2.4.42. ***(Serre)*** *Suppose S_0 is a Noetherian ring, and $S_\bullet$ is generated in degree 1. Let $\mathscr{F}$ be a finite type sheaf on Proj $S_\bullet$. Then there exist some N such that for all $n \geq N$, $\mathscr{F}(n)$ can be generated by a finite number of global sections.*

Sketch of the proof. Let $f \in S_\bullet$ with degree 1. We have $\mathscr{F}|_{D(f)} = \tilde{M}$ for some finitely generated $(S_\bullet[1/f])_0$-module M. If $m_1, \ldots, m_k$ generate M, then they generate $\mathscr{F}$ on $D(f)$ by viewing them as global sections on $\mathscr{F}|_{D(f)}$. We should somehow extend m_i to global sections on $\mathscr{F}$. But m_is for different f do not necessarily agree on overlaps of distinguished open sets $D(f)$ to be glued together. However, if we consider $f^N m_i$ for a sufficiently large N, then it will extend over another distinguished open set $D(g)$. The new poblem is that these various extensions may not agree on the overlaps, like $D(g) \cap D(g')$. Again we multiply both extensions by some $f^{N'}$ to become the same on the overlap. By compactness, we only need to consider a finite number of distinguished sets $D(f)$, so after a finite number of steps, by the aforementioned process, we can make various m_i to be compatible and glue them to get a finite number of global sections generating $\mathscr{F}$. □

Corollary 2.4.43. *Let $\mathscr{F}$ be a coherent sheaf on X, then:*

There exists $t \geq 0, d_1, \dots d_t \in \mathbb{Z}$ and a surjection

$$\bigoplus_{1\leq j\leq t} \mathcal{O}_X(d_j) \to \mathscr{F}$$

Proof. By the above theorem, suppose we have k global sections $s_1, ..., s_k$ of $\mathscr{F}(n)$ for some n generating $\mathscr{F}(n)$. Therefore we have a surjection

$$\bigoplus_1^k \mathcal{O}_X \to \mathscr{F}(n)$$
$$(a_1, \dots, a_k) \mapsto a_1 s_1 + \dots + a_k s_k$$

Tensoring with $\mathcal{O}_X(-n)$ gives the result, since tensoring by an invertible sheaf is exact. □

Theorem 2.4.44. *Let $\mathscr{F}$ be coherent sheaf on X, then:*

(i) $H^i(X, \mathscr{F}) = 0$ for $i \geq n+1$.

(ii) For any i, $H^i(X, \mathscr{F})$ is a finite dimensional R-module (finite dimensional vector space over K)

(iii) If $i > 0$, $H^i(X, \mathscr{F}(d)) = 0$ for all d large enough.

Proof. **(i)**: $X = P^n_R$. Clearly X is quasi-compact and separated. X can be covered by $n+1$ usual open affine sets $D(x_i)$. By lemma 2.4.29, we conclude that $H^i(X, \mathscr{F}) = 0$ for $i \geq n+1$.
(ii),(iii) : We prove 2,3 simultaneously by descending induction on i. For $i \geq n+1$, they are obviously true by (1). Suppose results hold for (i+1). By 2.4.43 there is a surjection $\bigoplus_{1\leq j\leq t} \mathcal{O}_X(d_j) \to \mathscr{F}$. Take $\mathscr{G}$ its kernel, which is coherent, to get a short exact sequence :

$$0 \to \mathscr{G} \to \bigoplus_{1\leq j\leq t} \mathcal{O}_X(d_j) \to \mathscr{F} \to 0 \tag{1}$$

Apply the cohomology functors to get a long exact sequence

$$H^i(X, \bigoplus_{1\leq j\leq t} \mathcal{O}_X(d_j)) \to H^i(X, \mathscr{F}) \to H^{i+1}(X, \mathscr{G}) \tag{2}$$

By induction hypothesis, $H^{i+1}(X, \mathscr{G})$ is a finite R-module. Therefore the middle term in the exact sequence is finite as well. Induction step if proved for (ii). By the corollary 2.4.38 twisting is an exact functor, hence (1) gives the exact sequence

$$0 \to \mathscr{G}(d) \to \bigoplus_{1\leq j\leq t} \mathcal{O}_X(d_j + d) \to \mathscr{F}(d) \to 0 \tag{3}$$

Apply the cohomology functors to get the long exact sequence

$$H^i(X, \bigoplus_{1\leq j\leq t} \mathcal{O}_X(d_j + d)) \to H^i(X, \mathscr{F}(d)) \to H^{i+1}(X, \mathscr{G}(d)) \tag{4}$$

By the induction hypothesis, $H^{i+1}(X, \mathscr{G}(d) = 0$ for $d \gg 0$.
By Theorem 2.4.40, $H^i(X, \bigoplus_{1\leq j\leq t} \mathcal{O}_X(d_j + d)) = 0$, for large d and $i > 0$. Consequently $H^i(X, \mathscr{F}(d)) = 0$ for $d \gg 0$. Induction step is proved for (iii).
Note that we can not get to the case $i = 0$, since it is not true that $H^0(X, \bigoplus_{1\leq j\leq t} \mathcal{O}_X(d_j + d)) = 0$ for large d. □

2.5 Complex analysis in several variables

All local complex function theory come from two famous theorems, Weierstrass Preparation theorem and Weierstrass Division theorem. The first one proved by Weierstrass while the later despite its name was proved by Stickelberger in 1887. These theorems imply e.g. the rings of germs of holomorphic functions are Noetherian. Our main reference for this section is [14].

2.5.1 Weierstrass theorems

In this section we use the notation $(z, w) = (z_1, \ldots z_{n-1}, w), z \in \mathbb{C}^{n-1}, w \in \mathbb{C}$. We will let O stand for the structure sheaf of $\mathbb{C}^n$ and $\mathcal{O}'$ for that of $\mathbb{C}^{n-1}$. The stalks $\mathcal{O}_0$ resp. $\mathcal{O}'_0$ are identified with the rings of convergent power series in $z_1, \ldots, z_{n-1}, w$ resp. in $z_1, \ldots, z_{n-1}$. The projection $\mathbb{C}^n \to \mathbb{C}^{n-1}$ induces an analytic monomorphism and we consider $\mathcal{O}'_0$, $\mathcal{O}'_0[w]$ of polynomials in w with coefficients in $\mathcal{O}'_0$ as subalgebras of $\mathcal{O}_0$.

We are going to state and prove Weierstrass Division theorem, first recall division theorem in a polynomial ring. For each polynomial ring $R[T]$ over a commutative ring R with identity and a polynomial $g = g_0 + g_1 T + \ldots + g_b T^b \in R[T]$ such that g_b is a unit in R, for every $f \in R[T]$ there exist uniquely determined polynomials $q, r \in R[T]$ such that

$$f = qg + r \quad \text{and} \quad \deg r < b$$

Because $\mathcal{O}'_0$ is a commutative ring the above theorem holds for the polynomial ring $\mathcal{O}'_0[w]$. Our goal is to prove analogues theorem for the ring $\mathcal{O}_0$, before that we need to define the degree of order of elements in $\mathcal{O}_0$. an element $g \in \mathcal{O}_0$ has order b in w, if $g(0, w) = w^b e(w)$ with a holomorphic unit e, i.e. $e(0) \neq 0$. If $g(0, w) \equiv 0$ we say it has infinite order. Expressing $g = \sum_{y=0}^{\infty} g_v w^v$ as a power series in w with coefficients in $\mathcal{O}'_0$, this condition is equivalent to

$$g_0(0) = \ldots = g_{b-1}(0) = 0 \text{ and } g_b(0) \neq 0$$

.

Theorem 2.5.1. ***(Weierstrass Division Theorem)*** *If $g \in \mathcal{O}_0$ has order b in w, then for every germ $f \in \mathcal{O}_0$ there exist a germ $q \in \mathcal{O}_0$ and a polynomial $r \in \mathcal{O}'_0[w]$ with $\deg r < b$ such that $f = qg + r$. The elements q, r are uniquely determined.*

Corollary 2.5.2. *With the above notations, let $g \in \mathcal{O}_0$ have order b in w. For every $f \in \mathcal{O}_0$, write*

$$f = qg + r, \; r = r_0 + r_1 w + \ldots + r_{b-1} w^{b-1} \in \mathcal{O}'_o[w]$$

define the map

$$\phi : \mathcal{O}_0 \to (\mathcal{O}'_0)^b$$

$$f \mapsto (r_0, \ldots, r_{b-1})$$

Then ϕ is an $\mathcal{O}'_0$-module epimoprhism with kernel $\mathcal{O}_0 g$. Hence we have an $\mathcal{O}'_0$-module isomorphism $\mathcal{O}_0 / \mathcal{O}_0 g \cong (\mathcal{O}'_0)^b$.

Definition 2.5.3. ***(Weierstrass polynomial)*** *A monic polynomial $w = w^b + a_1 w^{b-1} + \ldots + a_b \in \mathcal{O}_0'[w]$ is called a Weierstrass polynomial in w if $a_1(0) = \ldots = a_b(0) = 0$. Clearly w has finite order b.*

For one variable, a holomorphic function $f(z)$ near 0 can be written as $z^k w(z)$ where $w(0) \neq 0$, and k is the order of the zero of f at 0. Weierstrass Preparation Theorem generalizes this fact:

Theorem 2.5.4. ***(Weierstrass Preparation Theorem)****: If $g \in \mathcal{O}_0$ has finite order b in w, then there exist a uniquely determined Weierstrass polynomial $\omega \in \mathcal{O}_0'[w]$ of degree b and a unit $e \in \mathcal{O}_0$ such that $g = e\omega$. If $g \in \mathcal{O}_0'[w]$ then $e \in \mathcal{O}_0'[w]$.*

Note that the condition of g has finite order is crucial. g has finite order in w iff it is not identically zero on the w-axis. By a linear change of coordinates we can always assume a non-zero germ has finite order. More precisely:

Lemma 2.5.5. *if $g_1, \ldots g_k$ are finite number of non-zero germs, there exist linear coordinates $(z_1', \ldots z_{n-1}', w)$ such that each germ g_i has finite order in w.*

An important theorem which will be used several times is Hensel's lemma.

Lemma 2.5.6. ***(Hensel's lemma)*** *Let $\omega = \omega(z, w) = w^b + a_1(z) w^{b-1} + \ldots + a_b(z) \in \mathcal{O}_0'[w]$ be a monic polynomial of degree $b \geq 1$ such that $\omega(0, w) = (w - c_1)^{b_1} \ldots (w - c_t)^{b_c}$, where $c_1, \ldots, c_t$ are pairwise different complex numbers. Then there exist monic polynomials $\omega_1, \ldots, \omega_t \in \mathcal{O}_0'[w]$ of degree $b_1, \ldots, b_t$ such that $\omega = \omega_1 \cdot \ldots \cdot \omega_t$ and $\omega_j(0, w) = \left(w - c_j\right)^{b_j}$, $j = 1, \ldots, t$. The polynomials $\omega_1, \ldots, \omega_t$ are uniquely determined.*

Let B be a connected open subset of $\mathbb{C}^d$ and consider a monic polynomial

$$\omega(z, w) = w^b + a_1(z) w^{b-1} + \ldots + a_b(z) \in \mathcal{O}_B(B)[w]$$

of degree $b \geq 1$ with coefficients $a_1(z), \ldots, a_b(z) \in \mathcal{O}_B(B)$. Fix a point $y \in B$ and let $c_1, \ldots, c_t \in \mathbb{C}$ be the distinct roots of the polynomial $\omega(y, w)$, i.e. $\omega(y, w) = (w - c_1)^{b_1} \cdot \ldots \cdot (w - c_t)^{b_t}$ where $b_1, \ldots, b_t$ are positive integers with $b_1 + \ldots + b_t = b$. We set $x_j := \left(y, c_j\right) \in B \times \mathbb{C}, 1 \leq j \leq t$, and shortly write $\mathcal{O}_y$ resp. $\mathcal{O}_{x_j}$ instead of $\mathcal{O}_{B,y}$ resp. $\mathcal{O}_{B \times \mathbb{C}, x_j}$. We obtain natural algebra homomorphisms $\mathcal{O}_y[w] \to \mathcal{O}_{x_j}, j = 1, \ldots, t$. Thus, every polynomial $r \in \mathcal{O}_y[w]$ determines a germ $r_{x_j} \in \mathcal{O}_{x_j}$ at each point $x_j, j = 1, \ldots, t$. With these notations in mind:

Theorem 2.5.7. ***Theorem (Generalized Weierstrass Division Theorem)****: For any choice of germs $f_j \in \mathcal{O}_{x_j}, j = 1, \ldots, t$, there exist germs $q_j \in \mathcal{O}_{x_j}, 1 \leq j \leq t$, and a polynomial $r \in \mathcal{O}_y[w]$, with $\deg r < \deg \omega$ such that (1)*

$$f_j = \omega_{x_j} q_j + r_{x_j} \quad \text{for } j = 1, \ldots, t$$

The polynomial r and the germs $q_1, \ldots, q_t$ are uniquely determined by $f_1, \ldots, f_t$ and ω.

2.5.2 Complex spaces and Analytic sheaves

Definition 2.5.8. *let $D \subset \mathbb{C}^n$ be a connected open subset. Let $\mathcal{I}$ be finite type ideal sheaf of $\mathcal{O}_D$. It means that for every $x \in D$ there is an open neighborhood U_x and functions $f_1, \dots f_k \in \mathcal{O}(U_x)$ such that $\mathcal{I}(U_x) = \mathcal{O}(U_x)f_1 + \dots + \mathcal{O}(U_x)f_k$. Set $Y = Supp(\mathcal{O}_D/\mathcal{I})$ and $\mathcal{O}_Y = (\mathcal{O}_D/\mathcal{I})|_Y$. We call $(Y, \mathcal{O}_Y)$ a **complex analytic model**. Note that we have $Y \cap U_x = \{z \in U_x | f_1(z) = \dots = f_k(z) = 0\}$.*

Note that every finite set of holomorphic functions $f_1, \dots f_k \in \mathcal{O}(U)$ define a complex analytic model by setting $Y = \{z \in U | f_1(z) = \dots = f_k(z) = 0\}$ and $\mathcal{O}_Y = \mathcal{O}_U/(f_1, \dots, f_k)$.

Definition 2.5.9. *A **complex analytic space** is a locally ringed space $(X, \mathcal{O}_X)$ such that each point $x \in X$ has an open neighborhood U that $(U, \mathcal{O}_X|_U) = (Y, \mathcal{O}_Y)$ as locally ringed space for some complex analytic model $(Y, \mathcal{O}_Y)$.*

A morphism between analytic spaces is a morphism of underlying locally ringed spaces.

2.5.3 Algebraic properties of Analytic Stalks

In this section we state and prove some algebraic properties of an analytic stalk of the sheaf of holomorphic functions.

Let $X = (\mathbb{C}^n, \mathcal{O}_{\mathbb{C}^n})$ be the complex n-space equipped with the sheaf of germs of holomorphic functions. Let $O_{X,x}$ be the the stalk at x.

Lemma 2.5.10. *$O_{X,x} \cong \mathbb{C}\{z_1, \dots, z_n\}$, the algebra of convergent series in n variables, for all $x \in \mathbb{C}^n$*

Proof. A holomorphic function f defined over an open neighbourhood U of x can be represented by a power series convergent in U. Every two holomorphic functions that coincide on an open neighbourhood of x have the same power series in a possibly smaller neighborhood around x. Thus, $O_{X,x}$, defined as classes of holomorphic functions around x corresponds to $\mathbb{C}\{z_1, \dots, z_n\}$. □

Corollary 2.5.11. *For an analytic space $(X, \mathcal{O}_X)$, $O_{X,x} \cong \mathbb{C}\{z_1, \dots, z_n\}/\mathscr{I}$, for an ideal $\mathscr{I}$.*

Lemma 2.5.12. *$O_{X,x}$ is a local algebra with the unique maximal ideal $m = \{f | f(x) = 0\}$. Moreover $O_{X,x}/m = \mathbb{C}$.*

Theorem 2.5.13. *$O_{X,x}$ is a Noetherian ring which is UFD.*

Proof. We only prove the Noetherian property. It suffices to show the theorem for $X = \mathbb{C}^n$ and $x = 0$, since for arbitrary complex space X, $O_{X,x}$ is the quotient ring of some $O_{\mathbb{C}^n,x}$. We proceed by induction. For the case $n = 0$ it is nothing to prove. We use the following basic algebraic lemma.

Lemma 2.5.14. *A ring R is Noetherian iff each quotient ring R/rR is Noetherian for all $r \in R, r \neq 0$.*

Let $g \in \mathcal{O}_0, g \neq 0$. By corollary 2.5.2 we have an $\mathcal{O}'_0$-module isomorphism $\mathcal{O}_0/\mathcal{O}_0 g \cong (\mathcal{O}'_0)^b$ for some $b \geq 1$. By induction hypothesis $\mathcal{O}'_0$ and so $(\mathcal{O}'_0)^b$ are Noetherian. Hence $\mathcal{O}_0/\mathcal{O}_0 g$ is a Noetherian module over a Noetherian ring. Therefore, it is a Noetherian ring. □

Hilbert's Nullstellensatz theorem is a central tool in classical algebraic geometry. Its analytic version also holds. First we need some definition.

Definition 2.5.15. *If $f \in \mathcal{O}_{\mathbb{C}^n,0}$ is a germ represented by a holomorphic function $\widetilde{f} : U \to \mathbb{C}$, let $V_0(f)$ be the equivalence class of the set $\left\{z \in U | \widetilde{f}(z) = 0\right\}$ where two subsets $X, Y \subset \mathbb{C}$ are considered equivalent if $X \cap U = Y \cap U$ for some neighborhood U of 0. For each ideal $I \subset \mathcal{O}_{\mathbb{C}^n,0}$ let $V_0(I)$ denote $V_0(f_1) \cap \cdots \cap V_0(f_r)$ for some generators f_i. It is independent of a choice of the generators.*

For each subset $X \subset \mathbb{C}^n$ let $I_0(X) = \{f \in \mathcal{O}_{\mathbb{C}^n,0} | V_0(f) \supset X\}$. $I_0(X)$ is an ideal of $\mathcal{O}_{\mathbb{C}^n,0}$.

Theorem 2.5.16. ***(Analytic Nullstellensatz)*** *For each ideal $I \subset \mathcal{O}_{\mathbb{C}^n,0}$ we have*

$$\sqrt{I} = I_0(V_0(I))$$

Proof. See [16]. □

2.5.4 Stein spaces

An Stein space is an anlytic space which has sufficiently holomorphic functions. Stein spaces are the analogues of affine varieties or affine schemes in algebraic geometry. Our reference for this section is [15].

Definition 2.5.17. *Let $(X, \mathcal{O}_X)$ be an analytic space with separable topology, and $K \subset X$ be a compact subset. The* ***Holomorphic Convex hull*** *of K is $\hat{K}$:*

$$\hat{K} = \{x \in X| \, |f(x)| \leq \|f\|_K \text{ for all } f \in \mathcal{O}_X\}$$

Definition 2.5.18. *Let $(X, \mathcal{O}_X)$ be an analytic space. X is called a* ***stein space*** *if:*

(i) the topology of X is second countable.

(ii) for all compact subset $K \subset X$, $\hat{K}$ is also compact.

(iii) X is holomorphically separable, i.e. for $x \neq y \in X$, there is $f \in \mathcal{O}_X(X)$ such that $f(X) \neq f(y)$.

Remark 2.5.19. *A Stein sapce of positive dimesion is not compact, since it should have non-constant global holomorphic functions.*

Example 2.5.20. *(1) every closed analytic subspace of $\mathbb{C}^n$ is a Stein space.*

(2)Any non compact 1-dimensional analytic space with seperable toplogy is Stein.

Lemma 2.5.21. *Let $(X, \mathcal{O}_X)$ be an analytic space. If Y, Z are Stein subspaces, the so is $Y \cap Z$.*

3 Coherent sheaves on complex spaces

3.1 Oka's coherence theorem

This section closely follows chapter 2 of [14].

3.1.1 Introduction

Our main goal in this section is to prove the following deep theorem:

Theorem 3.1.1. ***(Oka's Coherence Theorem)*** *For every positive integer n, the structure sheaf $\mathcal{O}_{\mathbb{C}^n}$ (sheaf of germs of holomorphic functions on $\mathbb{C}^n$) is a coherent sheaf of rings.*

The problem of coherence of sheaf of holomorphic functions was first proposed by Cartan in 1944. Oka proved this theorem in 1948. We prove this theorem with a result called Coherence lemma and the Formal coherence criterion. The basic idea is to use Weierstrass Preparation theorem and Weierstrass Division theorem to reduce the proof of theorem to the case of one dimension less, then use induction.

Corollary 3.1.2. ***(Coherence of all structure sheaves $\mathcal{O}_X$)*** *The structure sheaf $\mathcal{O}_X$ of any complex space $(X, \mathcal{O}_X)$ is coherent.*

Proof. The problem being local we may assume that $(X, \mathcal{O}_X)$ is a complex model space in a domain D of $\mathbb{C}^n$, say $X = N(\mathscr{I})$ and $\mathcal{O}_X = (\mathcal{O}_D/\mathscr{I})\,|_X$, where $\mathscr{I}$ is an $\mathcal{O}_D$ -ideal of finite type. Since $\mathcal{O}_D$ is coherent by OKA's Theorem, $(\mathcal{O}_D/\mathscr{I})$ is also coherent. □

3.1.2 Plan of attack

The proof can be divide into 6 steps:

Step 1 :Finite maps
The Weierstrass preparation theorem reduces the study of zero set of a holomorphic function to the study of Weierstrass polynomial w. We have the projection map

$$(\text{zero set of } w) \to \mathbb{C}^{n-1} \text{ by } (z, w) \to z$$

It is a *finite map.*

A finite map f and its induced functor f_* of image sheaf are well behaved, e.g., f_* is an exact functor and there is a nice local representation of image sheaf. These results help us to give a simple proof of Oka coherence theorem. For these reasons we will develop some general properties of finite maps in the next section to prove the theorems.

Step 2: Weierstrass Maps

Let B be a domain(a connected open set) in $\mathbb{C}^d$ with coordinates $z = (z_1, \dots, z_d)$ and let $\omega_k = \omega_k\,(z, w_k) = w_k^{b_k} + a_{1\kappa}(z) w_k^{b_k - 1} + \ldots + a_{b_\kappa \kappa}(z) \in \mathcal{O}_B\,[w_\kappa]\,, \quad \kappa = 1, \dots, k$ be a monic polynomial of degree b_k in w_k with holomorphic coefficients. The zero set

$$A := N(\omega_1, \ldots, \omega_k) = \left\{(z,w) \in B \times \mathbb{C}^k : \omega_1(z, w_1) = \ldots = \omega_k(z, w_k) = 0\right\}$$

of $\omega_1, \ldots, \omega_k$ is a closed subset of $B \times \mathbb{C}^k$, in fact it is a closed complex subspace of $B \times \mathbb{C}^k$: $A = N(\mathscr{I}), \quad \mathcal{O}_A = \left(\mathcal{O}_{B\times\mathbb{C}^k}/\mathscr{I}\right) \mid A, \quad$ where $\mathscr{I} := \mathcal{O}_{B\times *}\omega_1 + \ldots + \mathcal{O}_{B\times\mathbb{C}^k}\omega_k$.

The projection $B \times \mathbb{C}^k \to B$ induces a holomorphic map $\pi : A \to B$. This map is surjective since $\omega_k(z, w_\kappa)$ has at least one root for any point $z \in B$(fundamental theorem of algebra); moreover every fiber $\pi^{-1}(z)$ is a finite set (consisting of at most $b_1 b_2 \cdot \ldots \cdot b_k$ different points). We call π the **Weierstrass map** defined by $\omega_1, \ldots, \omega_k$. We claim:

Theorem 3.1.3. *Every Weierstrass map $\pi : A \to B$ is closed, finite and an open map.*

Step 3: Weierstrass Isomorphism theorem

Now we prove the existence of a crucial sheaf isomorphism called The Weierstrass Isomorphism.

Consider the monic polynomial $\omega(z,w) = w^b + a_1(z)w^{b-1} + \ldots + a_b(z) \in \mathcal{O}_B(B)[w]$. We denote by $\pi : A \to B$ the induced Weierstrass map, recall that

$$\begin{aligned} A := N(\mathscr{I}) &= \{(z,w) \in B \times \mathbb{C} : \omega(z,w) = 0\} \\ \mathcal{O}_A := (\mathcal{O}_{B\times\mathbb{C}}/\mathscr{S}) \mid A & \quad \text{with } \mathscr{I} := \mathcal{O}_{B\times\mathbb{C}} \cdot \omega \end{aligned}$$

and that π is induced by the projection $B \times \mathbb{C} \to B$. We know that π is a finite map.

We are going to construct an $\mathcal{O}_B$ -module homomorphism

$$\bar{\pi} : \mathcal{O}_B^{\oplus b} \to \pi_*(\mathcal{O}_A)$$

and show that it is in fact an isomorphism. Let V be an open subset of B and let $s = (s_0, \ldots, s_{b-1}) \in \mathcal{O}_B^{\oplus b}(V)$. The polynomial $p_s = \sum_{\beta=0}^{b-1} s_\beta w^\beta \in \mathcal{O}_B(V)[w] \subset \mathcal{O}(V \times \mathbb{C})$ determines a section in $(\mathcal{O}/\mathscr{I})(V \times \mathbb{C})$ and therefore, by restriction to A, a section $s' \in (O/\mathscr{I})(V \times \mathbb{C}) \mid A = \mathcal{O}_A\left(\pi^{-1}(V)\right) = \pi_*(\mathcal{O}_A)(V)$. The map

$$\bar{\pi} : \mathcal{O}_B^{\oplus b}(V) \to \pi_*(\mathcal{O}_A)(V), \quad s \mapsto s'$$

is an $\mathcal{O}_B(V)$ -module homomorphism. The collection of these homomorphisms is compatible with restriction maps. Hence, we obtain an $\mathcal{O}_B$ -module homomorphism $\pi : \mathcal{O}_B^b \to \pi_*(\mathcal{O}_A)$, which we call the Weierstrass homomorphism of the Weierstrass map $\pi : A \to B$.

Theorem 3.1.4. *The Weierstrass homomorphism $\bar{\pi} : \mathcal{O}_B^{\oplus b} \to \pi_*(\mathcal{O}_A)$ is an $\mathcal{O}_B$-module isomorphism.*

Step 4: Coherence Lemma

We finally prove *Coherence Lemma* as an application of Weierstrass Isomorphism Theorem.

Theorem 3.1.5. ***(Coherence Lemma)*** *If $\mathcal{O}_B$ is a coherent sheaf then the sheaf $\mathcal{O}_A$ is also coherent.*

Step 5: Formal Coherence Criterion

The last step to prove Oka coherence theorem, is an abstract criteria to prove a sheaf is coherent.

Definition 3.1.6. ***(Hausdorff sheaf)*** *We say a sheaf $\mathscr{F}$ is Hausdorff if it is Hausdorff as a topological space, i.e, if $s \in \mathscr{F}_x$ and $s' \in \mathscr{F}_{x'}$, $s \neq s'$, there are open neighborhoods $x \in U$ and $x \in V$ and sections $f \in \mathscr{F}(U), f' \in \mathscr{F}(V)$ such that $f_x = s, f'_{x'} = s'$ and $f_y \neq f'_y$ for every $y \in U \cap V$.*

Sheaf of holomorphic function $\mathcal{O}_{\mathbb{C}^n}$ is Hausdorff because we can choose U, V to be open connected sets and use the Identity theorem in Complex analysis.

Lemma 3.1.7. ***(Formal Coherence Criterion)*** *Let $\mathscr{A}$ be a Hausdorff sheaf on X, such that all stalks $\mathscr{A}_x, x \in X$, are integral domains. Then $\mathscr{A}$ is coherent if the following condition is satisfied:*

(*) For any open set $V \subset X$ and any section $s \in \mathscr{A}(V)$, the sheaf of rings $\mathscr{A}_V / s\mathscr{A}_V$ is coherent at every point $x \in V$ where $s_x \neq 0$.

Step 6: Proof of OKA's Coherence Theorem

Proof. We proceed by induction on n. The case $n = 0$ is clear. Take $n > 0$ and assume that the statement of the theorem is true for any integer $k, 0 \leq k < n$. Since $\mathcal{O} := \mathcal{O}_{\mathbb{C}^n}$ is a Hausdorff sheaf and since all stalks $\mathcal{O}_{\mathbb{C}^n,z}$ are integral domains we only need to verify condition (*) of the Lemma: Let U be an open subset of $\mathbb{C}^n$ and let $g \in \mathcal{O}(U)$ be a section over U. We must show that the sheaf of rings $\mathcal{O}_U / g\mathcal{O}_U$ is coherent at all points $x \in U$ where $g_x \neq 0$.

Let x be such a point. We may assume $g(x) = 0$, for otherwise $g_x\mathcal{O}_x = \mathcal{O}_x$, so $(\mathcal{O}_U / g\mathcal{O}_U)_x = 0$. Since it is a sheaf of rings, it means $1 = 0$ in the stalk at x. Therefore there is a neighborhood $x \in U$ such that $(\mathcal{O}_U / g\mathcal{O}_U)(U) = 0$, which is obviously coherent.

choose coordinates (z, w) in $\mathbb{C}^n$ such that $x = 0 \in \mathbb{C}^n$ and that $g_x(0, w) \neq 0$. Then, by the Preparation Theorem, there exists a monic polynomial $\omega_x \in (\mathcal{O}_{\mathbb{C}^{n-1}})_0[W]$ and a unit $e \in (\mathcal{O}_{\mathbb{C}^{n-1}})_0$ such that $g_x = w_x e$. Therefore $g_x\mathcal{O}_x = \omega_x\mathcal{O}_x$. Therefore there is an open neighborhood of x such that $\mathcal{O}_U / g\mathcal{O}_U$ and

Choose a neighborhood $B \subset \mathbb{C}^{n-1}$ of $0 \in \mathbb{C}^{n-1}$ such that ω_x has a representative $\omega \in \mathcal{O}_B[w]$. We consider the corresponding Weierstrass map $\pi : (A, \mathcal{O}_A) \to (B, \mathcal{O}_B)$, where

$$A := \{(z, w) \in B \times \mathbb{C} : \omega(z, w) = 0\} \quad \text{and} \quad \mathcal{O}_A = (\mathcal{O}_{B\times\mathbb{C}} / \omega\mathcal{O}_{B\times\mathbb{C}})\,|_A$$

The sheaf $\mathcal{O}_B$ is coherent by induction hypothesis because B is an open set in $\mathbb{C}^{n-1}$,. Therefore by Coherence Lemma, the sheaf $\mathcal{O}_A$ is coherent.

We have the injection $i : A \to B \times \mathbb{C})$, the trivial injection $i_*\mathcal{O}_A = \mathcal{O}_{B\times\mathbb{C}} / \omega\mathcal{O}_{B\times\mathbb{C}}$ is coherent on $B \times \mathbb{C}$. Since $g_x\mathcal{O}_x = \omega_x\mathcal{O}_x$., there is an open neighborhood of x such that $\mathcal{O}_U / g\mathcal{O}_U$ and $\mathcal{O}_{B\times\mathbb{C}} / \omega\mathcal{O}_{B\times\mathbb{C}}$ coincide. Therefore $\mathcal{O}_v / g\mathcal{O}_U$ is coherent at x. □

3.1.3 Details of the proof

Finite maps

Definition 3.1.8. ***(Finite maps)*** *A continuous and closed map $f : X \to Y$ is called finite if each fiber $f^{-1}(y), y \in f(X)$, consists of a finite number of points.*

Clearly the composition $g \circ f : X \to Z$ of two finite maps $f : X \to Y, g : Y \to Z$ is finite again. If X is a closed subspace of Y the injection $X \to Y$ is finite. It's easy to see for each subset V of Y the induced map $f_{f-1}(V), v'(V) \to V$ is finite. The fundamental result for finite maps are the following local description of such maps.

Theorem 3.1.9. ***(Local Description of Finite Maps)****: Let X be a Hausdorff space and let $f : X \to Y$ be a finite map. Let $x_1, \dots, x_t$ be the distinct points of a fiber $f^{-1}(y), y \in f(X)$, and let $U_1', \dots, U_t'$ be pairwise disjoint open neighborhoods of $x_1, \dots, x_t$ in X. Then any neighborhood V' of y in Y contains an open neighborhood V of y with the following properties:*
1) $U_1 := f^{-1}(V) \cap U_1', \dots, U_t = f^{-1}(V) \cap U_t'$ are pairwise disjoint open neighborhoods of $x_1, \dots, x_t$ in X.
2) $f^{-1}(V) = \bigcup_{j=1}^{t} U_j$; in particular $f\left(U_j\right) \subset V$ for all j.
3) All of the induced maps $f_{U_j,V} : U_j \to V$ are finite.

A map $f : X \to Y$ is called finite at $x \in X$ if there exist open neighborhoods U resp. V of x resp. $f(x)$ with $f(U) \subset V$ such that the induced map $f_{U,V} : U \to V$ is finite. A continuous bijection $f : X \to Y$ which is finite at all points x of X is evidently closed and hence a homeomorphism.

Let $f : X \to Y$ a finite map. We are going to discuss about the functor f_* of image sheaves.With previous notation we have:

Theorem 3.1.10. *Let $f : X \to Y$ a finite map.*

(1) for each sheaf $\mathscr{S}$ of sets on X there are canonical bijections

$$(f_{U_j,V})_*(\mathscr{S}|_{U_j}) \xrightarrow{\sim} \mathscr{S}(U_j), \qquad f_*(\mathscr{S})(V) \xrightarrow{\sim} \prod_1^t \mathscr{S}(U_j). \tag{5}$$

$$f_*(\mathscr{S})_V \xrightarrow{\sim} \prod_1^t \left(f_{U_j,v}\right)_* (\mathscr{S}|_{U_j}), \quad f_*(\mathscr{S})_y \xrightarrow{\sim} \prod_1^t \mathscr{S}_{x_j} \tag{6}$$

(2) the functor f_* of image sheaf is exact.

So far we have studied general topological properties of finite maps. Now we apply this machinery to finite holomorphic maps $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$.

Corollary 3.1.11. *Let $f : X \to Y$ be a finite holomorphic map. Then for every $\mathcal{O}_X$ module $\mathscr{S}$ on X there is an $\mathcal{O}_Y(V)$ -module isomorphism*

$$f_*(\mathscr{S})|_V \xrightarrow{\sim} \prod_1^t \left(f_{U_j,V}\right)_* (\mathscr{S})$$

and an $\mathcal{O}_y$ -module isomorphism

$$f_*(\mathscr{S})_y \xrightarrow{\sim} \prod_1^l \mathscr{S}_{x_i}$$

The functor f_* is exact: If $\mathscr{S}' \to \mathscr{S} \to \mathscr{S}''$ is an exact $\mathcal{O}_X$ -sequence then $f_*(\mathscr{S}) \to f_*(\mathscr{S}) \to f_*(\mathscr{S}'')$ is an exact $\mathcal{O}_Y$ -sequence.

Proof of theorem 3.1.3

Proof. Weierstrass map is open finite closed. for closedness we need the following lemma from analysis:

Lemma 3.1.12. *Let $p_{\mathrm{vi}} = w^b + u_{1\mathrm{v}}w^{b-1} + \ldots + u_{bv} \in \mathbb{C}[w]$ be a sequence of monic polynomials of degree $b \geq 1$, assume that every sequence $u_{\beta 1}, u_{\beta 2}, \ldots$ has a limit $u_\beta \in \mathbb{C}, \beta = 1, \ldots, b$. Let $c_v \in \mathbb{C}$ be a root of p_v. Then there exists a sub-sequence of the sequence c_v which converges towards a root of the polynomial $p := w^b + u_1 w^{b-1} + \ldots + u_b \in \mathbb{C}[w]$.*

Lemma can be seen by proving the sequence c_v are bounded, hence have a convergent sub-sequence. Because of continuity the limit should be the root of p.

Closedness: Let M be closed in A and let y be a point in the closure of $\pi(M) \subset B$. Then there is a sequence $(y_v, c_{1v}, \ldots, c_{kv}) \subset M$, where $y_v \in \pi(M)$ converges to $y \in B$. Since all $a_{j\kappa}(z)$ are continuous functions we always have $\lim a_{jk}(y_v) = a_{jk}(y)$. Thus, by repeated application of the Lemma, there is a sub-sequence of the sequence $(c_{1v}, \ldots, c_{kv}) \in \mathbb{C}^k$ converging to a point $(c_1, \ldots, c_k) \in \mathbb{C}^k$ such that c_κ is a root of $\omega_\kappa(y, w_\kappa)$. Since M is closed in $B \times \mathbb{C}^k$ we have $(y, c_1, \ldots, c_k) \in M$ and consequently $y = \pi(y, c_1, \ldots, c_k) \in \pi(M)$.

Open map: Take a point $p = (q, c) \in A$, where $q_1 = \pi(p)$ and $c = (c_1, \ldots, c_k) \in \mathbb{C}^k$. Let U be a neighborhood of p in A, we may assume that $U = A \cap (\hat{B} \times W)$, where $\hat{B} \times W$ is an open product neighborhood of p in $B \times \mathbb{C}^k$

By Hensel's Lemma we can choose $\hat{B}$ in such a way that there exist monic polynomials $\hat{\omega}_\kappa, \tilde{\omega}_\kappa \in \mathcal{O}(B)[w_\kappa]$ with the following properties:

$$\omega_\kappa \mid \hat{B} \times \mathbb{C}^k = \hat{\omega}_\kappa \tilde{\omega}_\kappa, \hat{\omega}_\kappa(q, w) = (w - c_\kappa)^{n_{\kappa'}}, \quad 1 \leq \kappa \leq k$$

Now consider the closed complex subspace $\hat{A}$ of $\hat{B} \times \mathbb{C}^k$ defined by $\hat{\omega}_1, \ldots, \hat{\omega}_k$. Clearly $\hat{A} = N(\hat{\omega}_1, \ldots, \hat{\omega}_k) \subset A$. By construction p is the only point in the fiber over q of the projection $\hat{\pi} : \hat{A} \to \hat{B}$. Hence, there is a neighborhood $V \subset \hat{B}$ of q such that

$$\hat{A} \cap \left(V \times \mathbb{C}^k\right) \subset \hat{A} \cap (\hat{B} \times W) \subset U$$

Since clearly $\hat{\pi}\left(\hat{A} \cap \left(V \times \mathbb{C}^k\right)\right) = V$ we conclude $V \subset \pi(U)$. □

Proof of Weierstrass isomorphism 3.1.4

Proof. It is enough to show that every homomorphism of germs, $\bar{\pi}_y : \mathcal{O}_y^b \to \pi_*(\mathcal{O}_A)_y, y \in B$, is bijective. Since π is finite, every germ $g_y \in \pi_*(\mathcal{O}_A)_y$ corresponds uniquely, by Theorem 3.1.11, to a t -tuple $(g_1, \ldots, g_t) \in \prod_{j=1}^t \mathcal{O}_{A,x_j}$.

We have $\mathcal{O}_{A,x_j} \cong (\mathcal{O}/\mathscr{I})_{x_j} \cong \mathcal{O}_{x_j}/\mathscr{I}_{x_j}$, Therefore, $g_j \equiv f_j$ modulo $\mathscr{I}_{x_j}$ for germs $f_j \in \mathcal{O}_{x_j}, 1 \leq j \leq t$.
Now, by the Generalized Weierstrass Division Theorem, there exists a uniquely determined polynomial $r := \sum_{i=0}^{b-1} r_\beta w^\beta \in \mathcal{O}_y[w]$ such that $r_{x_j} \equiv f_j$ modulo $\mathscr{I}_{x_j}, 1 \leq j \leq t$. Thus, $g_j \equiv r_{x_j}$ modulo $\mathscr{I}_{x_j}, j = 1, \ldots, t$, and therefore $(r_0, \ldots, r_{b-1})$ is a uniquely determined element in $\mathcal{O}_y^b$ satisfying $\bar{\pi}_y\,(r_0, \ldots, r_{b-1}) = g_y$. □

Proof of Coherence Lemma 3.1.5

Proof. Since $\mathcal{O}_A$ is the structure sheaf we only have to prove the second condition of coherent sheaf, i.e. that for every open set $U \subset A$ and every choice of sections $s_1, \ldots, s_p \in \mathcal{O}_A(U)$, the kernel of the $\mathcal{O}_U$ -homomorphism

$$\varphi : \mathcal{O}_U^p \to \mathcal{O}_U, \quad (f_{1x}, \ldots, f_{px}) \mapsto f_{1x}s_{1x} + \ldots + f_{px}s_{px}, \quad x \in U$$

is of finite type at every point of U.
We first deal with the case $U = A$.

a) For every $\mathcal{O}_A$ -homomorphism $\varphi : \mathcal{O}_A^p \to \mathcal{O}_A$ the kernel φ is a finitely generated $\mathcal{O}_A$ -sheaf.

The Weierstrass isomorphism $\pi_*\,(\mathcal{O}_A) \cong \mathcal{O}_B^{\oplus b}$ tells us that all sheaves $\pi_*\left(\mathcal{O}_A^{\oplus p}\right) = \pi_*\,(\mathcal{O}_A)^{\oplus p}, 1 \leq p < \infty$, are $\mathcal{O}_B$ -coherent. Apply π_* to φ to get an $\mathcal{O}_B$ -homomorphism $\pi_*\left(\mathcal{O}_A^{\oplus p}\right) \to \pi_*\,(\mathcal{O}_A)$. Both of these are coherent, so its kernel. Since $\mathscr{K}_{\text{er}}\,\pi_*(\varphi) = \pi_*\,(\mathscr{K}_{er}\varphi)$, $\pi_*\,(\mathscr{K}_{er}\varphi)$ is $\mathcal{O}_B$ -coherent.

Thus each point $y \in B$ has a neighborhood $V \subset B$ with sections $t_1, \ldots, t_m \in \pi_*(\mathscr{K}er\varphi)(V)$ generating $\pi_*(\mathscr{K}er\varphi)|_V$. Since $\pi_*(\mathscr{K}er\varphi)|_V = (\mathscr{K}er\varphi)(\pi^{-1}(V))$, (t_i) generate $\mathscr{K}er\varphi$ over the open set $\pi^{-1}(V)$. So $\mathscr{K}er\varphi$ is a finitely generated $\mathcal{O}_A$ -sheaf.

General case) Now let $U \subset A$ be an open set and let $s_1, \ldots, s_p \in \mathcal{O}_A(U)$ be given sections. By Theorem (???? local discription of finte maps) there exists, to every point $x \in U$, a neighborhood $V \subset B$ of $\pi(x)$ such that $\pi^{-1}(V)$ is the union of pairwise disjoint neighborhoods $U_1, \ldots, U_t$ of the points of A over $\pi(x)$. Assume that $U_1 \subset U$ is a neighborhood of x. We extend $s_j|_{U_1}$ to a section $\hat{s}_j \in \mathcal{O}_A\left(\pi^{-1}(V)\right)$ by taking the zero cross section over $U_2, \ldots, U_t, 1 \leq j \leq p$. Then

$$(*) \quad \mathscr{K}er\ \hat{\varphi}\,|U_1 = \mathscr{K}er\ \varphi|\,U_1$$

where $\hat{\varphi} : \mathcal{O}_{\pi^{-1}(V)}^p \to \mathcal{O}_{\pi^{-1}(V)}$ resp. $\varphi : \mathcal{O}_{U_1}^p \to \mathcal{O}_{U_1}$ is induced by $\hat{s}_1, \ldots, \hat{s}_p$ resp. $s_1, \ldots, s_p$.
Consider the inducing map $\pi : \pi^{-1}(V) \to V$ and from a) we see that $\mathscr{K}_{er}\hat{\varphi}$ is a finitely generated $\mathcal{O}_{\pi^{-1}(V)}$ -sheaf. Hence (*) implies that $\mathscr{K}er\ \varphi|U_1$ is a finitely generated $\mathcal{O}_{U_1}$ -module. Since x was arbitrary $\mathscr{K}er\ \varphi$ is finitely generated. □

3.2 Cartan theorems A and B

Cartan's theorems A and B are two prominent results proved by Henri Cartan around 1951, concerning a coherent sheaf $\mathscr{F}$ on a Stein manifold. They are significant both as applied to several complex variables, and in the general development of sheaf cohomology. For proofs see [15].

Theorem 3.2.1. ***(Theorem A)*** *Let $(X, \mathcal{O}_X)$ be a stein space. Suppose $\mathscr{F}$ is a cohernet sheaf of X, then $H^0(X, \mathscr{F})$ generates $\mathscr{F}_x$ for all $x \in X$.*

Theorem 3.2.2. ***(Theorem B)*** *Let $(X, \mathcal{O}_X)$ be a stein space. Suppose $\mathscr{F}$ is a cohernet sheaf of X, then $H^i(X, \mathscr{F}) = 0$ for all $i \geq 1$.*

Remark 3.2.3. *These theorems are analogous to the theorem that higher cohomology groups for a coherent sheaf vanish for an affine scheme or variety. Therefore, Stein spaces are the corresponding objects of the affine schemes.*

3.3 Finiteness of sheaf cohomology groups of compact complex spaces

This section closely follows [15].

The following theorem is proved by Cartan and Serre in 1953.

Theorem 3.3.1. *Let $(X, \mathcal{O}_X)$ be a compact analytic space and $\mathscr{M}$ be an arbitrary coherent sheaf of $\mathcal{O}_X$-modules. Then all the spaces $H^q(X, \mathscr{M})$ are finite dimensional vector spaces over $\mathbb{C}$ for $q \geq 0$.*

In order to the prove the theorem we need to know about Frechet Spaces.

3.3.1 Frechet Spaces

Definition 3.3.2. *A **Frechet space** F is a vector space with a sequence of pseudonorms $\|.\|_p$ satisfying the following:*
(1) $\|a\|_p = 0$ for all p iff $a = 0$
(2) if (a_i) is a cauchy sequence in every pseudonorm, then there exist an $a \in F$ such that $\|a - a_i\|_p \to 0$ for all p.

Let $(X, \mathcal{O}_X)$ be an analytic space. We equip the the space global sections on X $(H^0(X, \mathcal{O}_X))$ with the **uniform convergence on compact set** topology as follows. Define the pseudonorms $\|.\|_K$ on all compact sets $K \subset X$ as:

$$\|f\|_K = sup\{|f(x)| x \in K\}$$

Now define a basis of open sets by the collection of $N(f, \epsilon, K)$ for $f \in \mathcal{O}_X, \epsilon > 0, K \subset X$ a compact set, where $N(f, \epsilon, K) = \{g \in \mathcal{O}_X | \, \|f - g\| < \epsilon_K\}$. Moreover a sequence of function $f_n \in H^0(U, \mathcal{O}_X|_U)$ converges to $f \in H^0(U, \mathcal{O}_X|_U)$ iff $\|f_n - f\|_K \to 0$ for all compact set K.

It can be seen that this topology is compact. In addition restriction map $H^0(U, \mathcal{O}_X|_U) \to H^0(V, \mathcal{O}_X|_V)$ is continuous for opens sets $V \subset U$. Moreover, $(H^0(X, \mathcal{O}_X))$ is a **Frechet space** with pseudonorms $\|.\|_K$. We can give similar topology to $(H^0(X, \mathscr{F}))$ for every coherent sheaf $\mathscr{F}$.

Theorem 3.3.3. *Let $(X, \mathcal{O}_X)$ be an analytic space. Let $\mathscr{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules. For any open set $U \subset X$, $(H^0(U, \mathscr{F}))$ can be given a topological structure of a Frechet space, such that restriction maps are continuous.*

Theorem 3.3.4. ***(Finitness theorem of Schwartz)*** *A locally compact Frechet space is finite dimensional.*

Finally, we need a functional analytic theorem proved by L. schwartz:

Theorem 3.3.5. *Let X, Y be two Frechet spaces, $u : X \to Y$ a surjective operator, and $v : X \to Y$ a compact operator. Then $(u+v)(X)$ is closed and $Y/(u+v)(X)$ is finite dimensional.*

3.3.2 Sketch of the proof

The case $q = 0$: by 3.3.3 equip $H^0(X, \mathscr{M})$ with the structure of a Frechet space. X is compact, so $X \subset \bar{X} \subset X$. The nontrivial point is that the induced map $r : H^0(X, \mathscr{M}) \to H^0(X, \mathscr{M})$ is a compact operator. Since r identity, $H^0(X, \mathscr{M})$ is a locally compact operator, hence finite dimensional by 3.3.5.

Suppose $q \geq 1$. It can be shown that there is a finite covering of X by $\mathfrak{U} = (U_i)$ and a refinement of $\mathfrak{U}$, namely $\mathfrak{V} = (V_i)$ which are Stein spaces. U_i and V_i are Stein spaces, hence their intersections $\bigcap_n U_{i_n}$ and $\bigcap_n V_{i_n}$ are also Stein spaces. By Cartan Theorem B, $H^q(\bigcap_n U_{i_n}, \mathscr{M}) = H^q(\bigcap_n V_{i_n}, \mathscr{M}) = 0$. Therefor Leray's theorem implies that we can compute sheaf cohomology by cech cohomology, i.e. for all $q \geq 1$.

$$\check{H}^q(\mathfrak{U}, \mathscr{M}) = \check{H}^q(\mathfrak{V}, \mathscr{M}) = H^q(X, \mathscr{M}) \tag{7}$$

We have $\check{C}^j(\mathfrak{U}, \mathscr{M}) = \prod_J H^0(\bigcap_J U_j, \mathscr{M})$. Using theorem 3.3.3, equip $\check{C}^j(\mathfrak{U}, \mathscr{M})$ with the product topology and make it to a Frechet space. Define the map $\phi : \check{C}^q(\mathfrak{U}, \mathscr{M}) \to \check{C}^q(\mathfrak{V}, \mathscr{M})$ by

$$\phi(f)(V_0 \cap \dots V_q) = f(U_0 \cap \dots U_q)|_{V_0 \cap \dots V_q}, \text{ for } f \in \check{C}^q(\mathfrak{U}, \mathscr{M}).$$

We have the following result about the map ϕ:

Theorem 3.3.6. *The map ϕ is a compact operator.*

Now the differential

$$\delta : \check{C}^q(\mathfrak{U}, \mathscr{M}) \to \check{C}^{q+1}(\mathfrak{U}, \mathscr{M})$$

is continuous. Hence $\ker\delta = \check{Z}^q(\mathfrak{U}, \mathscr{M})$ is closed, thus a Frechet space. The same is true for $\check{Z}^q(\mathfrak{V}, \mathscr{M})$. ϕ induce a map between sequences $C(\mathfrak{U}, \mathscr{M})$ and $C(\mathfrak{V}, \mathscr{M})$ which commutes with δ. Therefore, we have the compact operator

$$\phi : Z^q(\mathfrak{U}, \mathscr{M}) \to Z^q(\mathfrak{V}, \mathscr{M}).$$

By (1) restriction map $\check{H}^q(\mathfrak{U}, \mathscr{M}) = \frac{\check{Z}^q(\mathfrak{U}, \mathscr{M})}{\delta \check{C}^{q-1}} \to \frac{\check{Z}^q(\mathfrak{V}, \mathscr{M})}{\delta \check{C}^{q-1}} = \check{H}^q(\mathfrak{V}, \mathscr{M})$ is an isomorphism. Consequently the map

$$\theta : A = \check{Z}^q(\mathfrak{U}, \mathscr{M}) \bigoplus \check{C}^{q-1}(\mathfrak{V}, \mathscr{M}) \to \check{Z}^q(\mathfrak{V}, \mathscr{M})$$
$$\theta(f, g) = \phi(f) + \delta(g)$$

is surjective.

Now set $u = \theta$ and $v = -(\phi \oplus 0)$ in theorem 3.3.5. We get that $u + v = \delta$. By the theorem, $\frac{\check{Z}^q(\mathfrak{V},\mathscr{M})}{\delta(A)} = \check{H}^q(\mathfrak{V},\mathscr{M}) = H^q(X,\mathscr{M})$ is finite dimensional.

4 GAGA theorems

4.1 Analytification

In this section we associate an analytic space to a algebraic variety or a scheme. We assume our variety or scheme is defined over $\mathbb{C}$. The basic idea is as follows:

Roughly speaking, an algebraic variety over $\mathbb{C}$ is defined as zero set of some finite number of complex polynomials. Polynomials are continuous and in fact holomorphic. Therefore, a complex structure is induced on the zero sets of polynomials. Moreover, The usual complex topology is *finer* than Zarisky topology. A Regular function which is defined as a quotient of two polynomials is also holomorphic.

Corollary 4.1.1. *(1) The usual topology on $\mathbb{C}^n$ is finer than Zarisky topology.*
(2) A regular function $f : U \to V$ for two locally-closed subsets of $\mathbb{C}^n$ and $\mathbb{C}^m$ is holomorphic.
(3) A biregular isomorphism f is also an analytic isomorphism.

Since the construction of varieties differ from Schemes, we treat each of them separately.

4.1.1 Algebraic variety

First suppose X is an affine variety over the field of $\mathbb{C}$.

There are polynomials $f_1, \ldots, f_k$ on an open set $U \subset \mathbb{C}^n$ such that $X = V(f_1, \ldots, f_k)$. According tho the corollary, X can be given a usual complex topology and an analytic sheaf $\mathcal{O}_{C^n}(U)/(f_1, \ldots, f_k)$. Hence X can be seen as a complex space.

Suppose X is variety over the field of $\mathbb{C}$. By definition X can be covered by open sets (U_i) with algebraic charts $\phi : U_i \to V$ where U_i is an open Zarisky set and V is an affine variety. See V as a complex space. By corollary ϕ is holomorphic. ϕ^{-1} gives U_i a complex structure. It can be easily seen that complex structures on U_i agree on overlaps and we can glue them to get an analytic structure on X. It is clear from the definition of complex spaces that X becomes an analytic space, namely X^h .

The above construction is clearly functorial.

Remark 4.1.2. *If a variety X over $\mathbb{C}$ is nonsingular, with the above complex structure X^h becomes a complex manifold. This is because zero becomes a regular value for a holomorphic function and its zero set can be made into a complex manifold.*

Remark 4.1.3. *For the case of varieties, X and X^h are equal as sets, they only differ in their topology and sheaf structures.*

Corollary 4.1.4. *There is a bijective map of ringed space $h : X^h \to X$. In fact, it is a morphism of locally ringed space.*

The complex topology on the product of two complex spaces is the product topology, while the topology on the product of two algebraic varieties is the Zarisky topology induced by the Segre embedding, which is not the same as the product topology. However, the analytification process is well-behaved with respect to the product:

Theorem 4.1.5. *If X, Y are two algebraic varieties, then $(X \times Y)^h \simeq X^h \times Y^h$.*

Proof. First note that direct product exists in the category of complex spaces. For the topology it is suffices to check this locally, so assume X and Y are affine. Suppose $X = V(f_1, \ldots, f_N)$ for the affine space A^n with coordinates $k[x_1, ..., x_n]$ and $Y = V(g_1, \ldots, g_M)$ for the affine space A^m with coordinates $k[y_1, ..., y_m]$. Then $X \times Y = V(f_1, ..., f_N, g_1, ..., g_M)$ in A^{n+m}. The topology of $\mathbb{C}^{n+m}$ is equivalent to the product topology $\mathbb{C}^n \times \mathbb{C}^m$. Hence the open sets of the form $(U \times V) \cap V(f_1, ..., f_N, g_1, ..., g_M)$ for open sets $U \in \mathbb{C}^n, V \in \mathbb{C}^m$ constitute a base for the topology of $(X \times Y)^h$. However, since f_i and g_j have different coordinates we have $(U \times V) \cap V(f_1, ..., f_N, g_1, ..., g_M) = U \cap V(f_1, ..., f_N) \times V \cap V(g_1, ..., g_M)$ which forms a base for $X^h \times Y^h$. Similarly, the sheaf on $(X \times Y)^h$ is of the form $\mathcal{O}_D/(f_1, \ldots, f_N, g_1, \ldots, g_M)$ for the open set $D \subset \mathbb{C}^{n+m}$, where $\mathcal{O}$ stands for the sheaf of holomorphic functions and by viewing f_i, g_i as complex polynomials. If f is a polynomial on U, it can be seen as a function on $U \times V$ by viewing it as $f \circ \pi_1$ where π_1 is projection of $U \times V$ to U. From the construction of the product of two complex spaces, it follows that locally, the sheaf on $X^h \times Y^h$ is of the form $\mathcal{O}_{U \times V}/(f_1 \circ \pi_1, \ldots, f_N \circ \pi_1, g_1 \circ \pi_2, \ldots, g_M \circ \pi_2)$. Now since the topology of $\mathbb{C}^{n+m}$ is the same as the product topology, the result immediately follows.

$\square$

4.1.2 Finite type Schemes

Let $X = (Spec\ A)$ an affine scheme, where A is a finitely generated $\mathbb{C}$-algebra. Therefore, $A = \mathbb{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_k)$ for some complex polynomials $f_1, \ldots, f_k$ in $\mathbb{C}^n$. Set $X^h = V(f_1, \ldots, f_k)$ and make it to a complex space. By Hilbert's Nullstellensatz, the points $x \in X^h$ correspond to maximal ideals m_x of A. Therefor, there is a map $\phi : X^h \to X,\ x \mapsto m_x$.

Lemma 4.1.6. *(1) With Zariski-topology on X and usual toplogy on X^h, ϕ is continuous.*

(2) $\phi^\sharp : \mathcal{O}_X \to \phi_* \mathcal{O}_{X^h}$ is a morphism of sheafs.

Proof. Take a distinguished open set $D_a, a \in A = \mathbb{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_k)$. $\phi^{-1}(D_a) = \{x \in X^h | a \notin m_x\} = \{a(x) \neq 0\}$ which is clearly open in usual topology. Viewing $\frac{r}{a^i}$ as a holomorphic function for $a, r \in \mathbb{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_k)$ we get the obvious sheaf morphism $\mathcal{O}_X(D_a) = \{\frac{r}{a^i}\} \to \phi_* \mathcal{O}_{X^h}(D_a) = \mathcal{O}_{X^h}(\{a(x) \neq 0\})$.

Corollary 4.1.7. *The map $\phi : (Spec\ A)^h \to (Spec\ A)$ is morphism of ringed space. In fact, it is a morphism of locally ringed space(next section).*

Now suppose X is a scheme of finite type over $\mathbb{C}$. It means that X can be covered by $(U_i) = (Spec\ A_i)$, where A_i is a finitely generated $\mathbb{C}$-algebra. By above construction make (U_i) into a complex space (U_i'). With glueing data of (U_i), we can glue (U_i') together to get a complex space X^h.

The construction is functorial.

Corollary 4.1.8. *The map $\phi : X^h \to X$ is a morphism of ringed space.*

Remark 4.1.9. *In contrast to the varieties, as sets, X and X^h are different. X^h has fewer points, only the closed points of X. $\phi : X^h \to X$ is bijective on X^h to the closed points of X. However, X^h has a finer topology than the topology induced on the closed points of X.*

Remark 4.1.10. *Note that we have to maps: (1) $\phi : X^h \to X$ between them as locally ringed spaces, (2) $X \mapsto X$ which is the map of analytification functor.*

Remark 4.1.11. *$\phi : X^h \to X$ does two things: first it weakens the topology to Zarisky topology, second it adds additional points in the case of schemes.*

Example 4.1.12. *If $X = \mathbb{P}^n_{\mathbb{C}}$ as an algebraic variety (or scheme), $X^h = \mathbb{P}^n_{\mathbb{C}}$ as the complex manifold.*

4.1.3 Relationship between stalks

Let X be a projective algebraic variety or a finite type scheme on $\mathbb{C}$.

We have the canonical morphism $\theta : \mathcal{O}_{X,x} \to \mathcal{O}_{X^h,x}$ by viewing regular functions as holomorphic ones. Map θ is a local morphism, sending the maximal ideal of $\mathcal{O}_{X,x}$ to that of $\mathcal{O}_{X^h,x}$. By equipping these local rings with m-adic topology, θ becomes continuous. Therefore, we can extend it to a homomorphism $\hat{\theta} : \hat{\mathcal{O}}_{X,x} \to \hat{\mathcal{O}}_{X^h,x}$ of their completion.

The following theorem is one of the main ingredients in the proof of GAGA theorems.

Theorem 4.1.13. *The map $\hat{\theta} : \hat{\mathcal{O}}_{X,x} \to \hat{\mathcal{O}}_{X^h,x}$ is bijective. In other words, the stalks $\mathcal{O}_{X,x}$ and $\mathcal{O}_{X^h,x}$ have the same completion.*

Proof. First suppose $X = \mathbb{C}^n$. $\mathcal{O}_{X,x} = \mathbb{C}[z_1, \dots, z_n]_{m_x}$ for a maximal ideal $\mathfrak{m}$ and $\mathcal{O}_{X^h,x} = \mathbb{C}\{z_1, \dots, z_n\}$. Completion of the first one is $\mathbb{C}[[z_1, \dots, z_n]]$ because of lemma 2.2.10, completion of the second one is obviously $\mathbb{C}[[z_1, \dots, z_n]]$.

With the assumption of $X = \mathbb{C}^n$, let Y be Zarisky locally closed set of X. Let $\mathscr{I}_x(Y)$ be the ideal of $\mathcal{O}_{X,x}$ generated from function f whose restriction to Y is zero in a Zarisky neighborhood of x. Y can be viewed as a closed analytic subspace of X^h. Similarly let $\mathscr{A}_x(Y)$ be the ideal generated by $f \in \mathcal{O}_{X^h,x}$ whose restriction to Y is zero in a neighborhood of x. Obviously under θ, $\mathscr{I}_x(Y)$ can be viewed as a subring of $\mathscr{A}_x(Y)$. The following lemma is the essence of the proof the general case.

Lemma 4.1.14. *The ideal $\mathscr{A}_x(Y)$ is generated by $\theta(\mathscr{I}_x(Y))$.*

Proof. Under θ we identify $\mathcal{O}_{X,x}$ as a subring of $\mathcal{O}_{X^h,x}$. Let I be the ideal in $\mathcal{O}_{X^h,x}$ generated by the subring $\mathscr{I}_x(X)$. We should prove that $I = \mathscr{A}_x(X^h)$. We choose an arbitrary $f \in \mathscr{A}_x(X^h)$ and show that $f \in I$. From the Analytic Nullstellensatz theorem 2.5.16, we have $\sqrt{I} = I_0(V_0(I)) = I_0([Y])$. Since $f \in I_0([Y])$ it follows that there exist an integer $r \geq 0$ such that $f^r \in I$. Since we have proved 4.1.13 for the the case $\mathbb{C}^n$, we get

$$f^r \in I\hat{\mathcal{O}}_{X^h,x} = \mathscr{I}_x(Y)\hat{\mathcal{O}}_{X^h,x} = \mathscr{I}_x(Y)\hat{\mathcal{O}}_{X,x}.$$

Let Y_i be irreducible componenet of Y passing through x. Denote $p_i = \mathscr{I}_x(X_i)$, which they are prime ideals. Then we have $\mathscr{I}_x(X) = \bigcap p_i$. So $\mathscr{I}_x(X)$ is an intersection of prime ideals. By a theorem of Chevalley (see [2]), $\mathscr{I}_x(Y)\hat{\mathcal{O}}_{X,x}$ is also an intersection of prime ideals. So $f \in \mathscr{I}_x(Y)\hat{\mathcal{O}}_{X,x}$. We have proved before that $\mathcal{O}_{X,x}$ is a Noetherian local ring. Hence by 2.2.20, $\mathcal{O}_{X,x} \to \hat{\mathcal{O}}_{X^h,x}$ is faithfully flat. By the faithfully flat properties we have $I\hat{\mathcal{O}}_{X^h,x} \cap \mathcal{O}_{X,x} = I$. Therefore $f \in I$.

□

Now we prove the general case of the theorem. Since the question is local, we may assume X is an affine subvariety of an open set U. We have

$$\mathcal{O}_{X,x} = \mathcal{O}_{U,x}/\mathscr{I}_x(X)\mathcal{O}_{X^h,x} = \mathcal{O}_{U^h,x}/\mathscr{A}_x(X)$$

From the affine case of the theorem, $\hat{\theta} : \hat{\mathcal{O}}_{U,x} \to \hat{\mathcal{O}}_{U^h,x}$ is bijective. Since U is affine, from the above lemma, by substituting X as Y and U as the affine space, we know that $\mathscr{A}_x(X) = \theta(\mathscr{I}_x(X))\mathcal{O}_{U^h,x}$. Now the theorem immediately follows from the theorem 2.2.22.

□

Theorem 4.1.15. *The map $\theta : \mathcal{O}_{X,x} \to \mathcal{O}_{X^h,x}$ is a faithfully flat morphism.*

Proof. We have $\mathcal{O}_{X,x} \xrightarrow{\theta} \mathcal{O}_{X^h,x} \xrightarrow{i} \hat{\mathcal{O}}_{X^h,x} = \hat{\mathcal{O}}_{X,x}$ by theorem 4.1.13. i and $i \circ \theta$ are faithfully flat by theorem 2.2.20. Then θ is faithfully flat by Theorem 2.2.21. □

4.1.4 Corresponding analytic sheaves

Let $(X, \mathcal{O}_X)$ be the algebraic variety over $\mathbb{C}$ or a finite type scheme over $\mathbb{C}$, and $(X^h, \mathcal{O}_{X^h})$, the associated analytic space.

Definition 4.1.16. *we have the morphism of locally ringed space $h : X^h \to X$. For any sheaf $\mathscr{F}$ on X, we have a sheaf $\mathscr{F}^h = h^*\mathscr{F}$ on X^h which we call the* ***corresponding analytic sheaf****.*

Lemma 4.1.17. *The following identities hold.*

(1) $(h^{-1}\mathscr{F})_x = \mathscr{F}_x$.
(2) $(\mathcal{O}_X)^h = \mathcal{O}_{X^h}$.
(3) $(\mathscr{F}^h)_x = (\mathscr{F}_x) \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X^h,x}$.

Theorem 4.1.18. *$\mathscr{F}^h$ is an exact functor.*

Proof. Suppose $\mathscr{F} \to \mathscr{F}' \to \mathscr{F}''$ is an exact sequence. By definition, $\mathscr{F}^h = h^*\mathscr{F} = h^{-1}\mathscr{F} \otimes \mathcal{O}_{X^h}$. By Theorem 4.1.15, $\mathcal{O}_{X,x} \to \mathcal{O}_{X^h,x}$ is flat. Since h^{-1} and tensoring with a flat module is exact, the sequence

$$h^{-1}\mathscr{F} \otimes \mathcal{O}_{X^h,x} \to h^{-1}\mathscr{F}' \otimes \mathcal{O}_{X^h,x} \to h^{-1}\mathscr{F}'' \otimes \mathcal{O}_{X^h,x}$$

is exact. Exactness on stalks gives exactness of sheaves. □

By 3.1.2 we know that structure sheaf of any complex space is coherent, so $\mathcal{O}_{X^h}$ is coherent.

Theorem 4.1.19. *If $\mathscr{F}$ is a coherent algebraic sheaf, $\mathscr{F}^h$ is a coherent analytic sheaf.*

Proof. Let $x \in X$. By lemma 2.1.4 there is an open set $x \in V \subset X$(in Zarisky topology) and an exact sequence

$$\mathcal{O}_X^m|_V \to \mathcal{O}_X^n|_V \to \mathscr{F}|_V \to 0.$$

With the help of above results and applying the exact functor $\mathscr{F}^h$ we would get an exact sequence

$$\mathcal{O}_{X^h}^m|_V \to \mathcal{O}_{X^h}^n|_V \to \mathscr{F}^h|_V \to 0$$

V is an open set in the usual topology(it is finer than Zarisky topology), hence an open neighborhood of x in X^h. Since $\mathcal{O}_{X^h}$ is coherent by lemma, $\mathscr{F}^h$ is also coherent by 2.1.7. □

4.1.5 Homomorphisms induced on cohomology

We have the morhism of ringed spaces $h : X^h \to X$. Since $\mathscr{F}^h = h^*\mathscr{F}$, We have the natural homomorphisms on cohomolgy groups:

$$h^i : H^i(X, \mathscr{F}) \to H^i(X^h, \mathscr{F}^h)$$

The map h^i is functorial on with respect to sheaf $\mathscr{F}$, in other words for any morphism $\mathscr{F} \to \mathscr{G}$ of sheaves on X the following dagram commutes:

$$\begin{array}{ccc} H^i(X, \mathscr{F}) & \xrightarrow{h^i} & H^i(X^h, \mathscr{F}^h) \\ \downarrow f & & \downarrow f \\ H^i(X, \mathscr{G}) & \xrightarrow{h^i} & H^i(X^h, \mathscr{G}^h) \end{array}$$

where f is the induced morhism by cohomology functors.

Lemma 4.1.20. *We can compute both cohomology groups $H^i(X, \mathscr{F})$ and $H^i(X^h, \mathscr{F}^h)$ for coherent sheaves with the cech cohomology using an affine cover.*

Proof. This is because of **Leray** theorem 2.3.18. We saw this for $H^i(X, \mathscr{F})$ in 2.4.28. In addition, for an affine scheme $Y = Spec(\mathbb{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_k))$, we have $Y^h = V(f_1, \ldots, f_k)$ which is a Stein space, since every analytic closed subspace of $\mathbb{C}^n$ is a Stein space. Moreover, by Cartan theorem B, $\mathscr{F}^h$ is acyclic on Y^h. Therefore the conditions of Leray theorem is satisfied. □

Lemma 4.1.21. *if $\mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ is an exact sequnce of coherent sheaves on X, then the diagram*

$$\begin{array}{ccc} H^i(X, \mathscr{H}) & \xrightarrow{h^i} & H^{i+1}(X, \mathscr{F}) \\ \downarrow f & & \downarrow f \\ H^i(X^h, \mathscr{H}^h) & \xrightarrow{h^i} & H^{i+1}(X^h, \mathscr{F}^h) \end{array}$$

commutes.

Sketch of the proof. Take a cover by open affine sets. Use cech cohomology and the definition of differential. It is not hard but a boring computation. □

4.2 Statement of GAGA theorems

Let $X = \mathbb{P}^n$ as a protective variate or a scheme.

Theorem 4.2.1. ***(GAGA THM 1).*** *For any coherent sheaf $\mathscr{F}$ on X, the natural morphism*

$$h^i : H^i(X, \mathscr{F}) \to H^i(X^h, \mathscr{F}^h)$$

is an isomorphism for each i.

Theorem 4.2.2. ***(GAGA THM 2).*** *If $\mathscr{F}, \mathscr{G}$ are coherent sheaves on X, then the natural map*

$$Hom_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G}) \to Hom_{\mathcal{O}_X^h}(\mathscr{F}^h, \mathscr{G}^h)$$

is an isomorphism. In other words, $\mathscr{F}^h$ is a fully-faithful functor.

Theorem 4.2.3. ***(GAGA THM 3).*** *For every coherent sheaf $\mathscr{M}$ on X^h, there exist a unique(up to isomorphism) cohernet sheaf $\mathscr{F}$ on X such that $\mathscr{F}^h \cong \mathscr{M}$. That means $\mathscr{F}^h$ is essentially surjective.*

Corollary 4.2.4. *There is an equivalence of categories between the category of coherent sheaves on X and the category of coherent sheaves on X^h.*

4.3 Proof of theorem 1

Step 1: $\mathscr{F} = \mathcal{O}_X$

We first prove the claim for the structure sheaf $\mathcal{O}_X$. We know that it is a coherent sheaf of rings.

$\mathcal{O}_X^h = \mathscr{H}$ the sheaf of holomorphic functions. For $i = 0$, $H^0(X, \mathcal{O}_X) = \Gamma(X, \mathcal{O}_X) = \mathbb{C}$. Also $H^0(X^h, \mathscr{H}) = \Gamma(X^h, \mathscr{H}) = \mathbb{C}$ because of Liouville's theorem that the only holomorphic functions on a compact space is constant functions.

For $i > 0$, by 2.4.40 we now that $H^i(X, \mathcal{O}_X) = 0$.

By Dolbeault theorem on cohomolgy, $H^i(X^h, \mathscr{H}) = 0$. So the theorem is true for the sheaf $\mathcal{O}_X$.

Step 2: $\mathscr{F} = \mathcal{O}(n)$

Now we prove the theorem for twisting sheaves $\mathcal{O}(n), n \in \mathbb{Z}$. We argue by induction on r =dimension$X$.
The case $r = 0$ is trivial. For $r > 0$, let $x_1, \ldots x_r$ the homogeneous coordinates. Let $\mathbb{H} \cong \mathbb{P}^{r-1}$ the hyperplane defined by setting $x_r = 0$, therefore $\dim(H) < r$.

Apply the twisting functor $\mathscr{F}(n)$ to the exact sequence 2.4.39 to get the exact sequence

$$0 \to \mathcal{O}(n-1) \to \mathcal{O}(n) \to \mathcal{O}_{\mathbb{H}}(n) \to 0$$

for each $n \in \mathbb{Z}$. By applying cohomology functors and the exact functor $\mathscr{F}^h$ we get a commutative diagram

$$\begin{array}{ccccccccc} H^i(X, \mathcal{O}_{\mathbb{H}}(n-1)) & \longrightarrow & H^i(X, \mathcal{O}(n-1)) & \longrightarrow & H^i(X, \mathcal{O}(n)) & \longrightarrow & H^i(X, \mathcal{O}_{\mathbb{H}}(n)) & \longrightarrow & H^{i+1}(X, \mathcal{O}(n-1)) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^i(X^h, \mathcal{O}_{\mathbb{H}}(n-1)^h) & \to & H^i(X^h, \mathcal{O}(n-1)^h) & \to & H^i(X^h, \mathcal{O}(n)^h) & \to & H^i(X^h, \mathcal{O}_{\mathbb{H}}(n)^h) & \to & H^{i+1}(X^h, \mathcal{O}(n-1)^h) \end{array}$$

We have the natural isomorphisms

$$H^i(X, \mathcal{O}_{\mathbb{H}}(n)) \cong H^i(\mathbb{H}, \mathcal{O}_{\mathbb{H}}(n)),$$
$$H^i(X^h, \mathcal{O}_{\mathbb{H}}(n)^h) \cong H^i(\mathbb{H}^h, \mathcal{O}_{\mathbb{H}}(n)^h).$$

Also, by the inductive hypothesis we know the natural morphisms $H^i(\mathbb{H}, \mathcal{O}_{\mathbb{H}}(n)) \to H^i(\mathbb{H}^h, \mathcal{O}_{\mathbb{H}}(n)^h)$ are isomorphisms for all $i \geq 0$. If the theorem is true for $\mathcal{O}(n-1)$, it is also true for $\mathcal{O}(n-1)$, thanks to the five lemma applied to the above diagram. Since the theorem is true for $\mathcal{O}(0) = \mathcal{O}_X$ (previous step), theorem is true for all $\mathcal{O}(n)$.

Step 3: arbitrary coherent sheaf $\mathscr{F}$

Finally we prove the general case. We argue by **descending induction** on i. For $q \geq 2r$, $r = dim X$, we have $H^i(X, \mathscr{F}) = H^i(X^h, \mathscr{F}^h) = 0$ by 2.3.16 and 4.1.20.

Now suppose the theorem is true for $i+1$. By Theorem 2.4.43 there is a surjection $\mathscr{G} = \bigoplus_{i=1}^k \mathcal{O}(j) \to \mathscr{F}$. Let $\mathscr{K}$ the kernel of this map then we have an exact sequence of coherent sheaves

$$0 \to \mathscr{K} \to \mathscr{G} \to \mathscr{F} \to 0$$

Again by taking cohomology we get a commutative diagram

$$\begin{array}{ccccccccc} H^i(X, \mathscr{K}) & \longrightarrow & H^i(X, \mathscr{G}) & \longrightarrow & H^i(X, \mathscr{F}) & \longrightarrow & H^{i+1}(X, \mathscr{K}) & \longrightarrow & H^{i+1}(X, \mathscr{G}) \\ \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_2} & & \downarrow{\scriptstyle h_3} & & \downarrow{\scriptstyle h_4} & & \downarrow{\scriptstyle h_5} \\ H^i(X^h, \mathscr{K}^h) & \longrightarrow & H^i(X^h, \mathscr{G}^h) & \longrightarrow & H^i(X^h, \mathscr{F}^h) & \longrightarrow & H^{i+1}(X^h, \mathscr{K}^h) & \longrightarrow & H^{i+1}(X^h, \mathscr{G}^h) \end{array}$$

Morphisms h_4, h_5 are isomorphisms by the hypothesis. By **Step 2**, h_2 is also an isomorphism. By Five Lemma h_3 is an epimorphism. Since $\mathscr{F}$ was arbitrary the same is true for $\mathscr{K}$. Therefore h_1 is an epimorphism. By Five Lemma h_3 is an isomorphism too, proving the induction for i.

4.4 Proof of therem 2

Let $\mathscr{F}, \mathscr{G}$ be coherent sheaves on X. Consider the Hom sheaf $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G})$.

Lemma 4.4.1. *There is a canonical morphism*

$$\mathrm{i} : \mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G})^h \to \mathscr{H}om_{\mathcal{O}_{X^h}}(\mathscr{F}^h, \mathscr{G}^h)$$

Proof. This is a general property of pull back and hom sheaf. We have the map of locally ringed spaces $h : X^h \to X$. There is a natural map $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F},\mathscr{G}) \to h_*\mathscr{H}om_{\mathcal{O}_{X^h}}(\mathscr{F}^h,\mathscr{G}^h)$. Using the adjunction with h^* we get the required map. Intuitively, we get the map by viewing algebraic morphism on a Zarisky open set as an analytic map over the corresponding set on X^h. $\square$

Lemma 4.4.2. *the map* i *is an isomorphism.*

Proof. It is enough to prove the isomorphism on stalks. Because $\mathscr{F}, \mathscr{F}^h$ are coherent, by Lemma 2.1.13, for each $x \in X$ we have the natural isomorphisms

$$(\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F},\mathscr{G}))_x \cong \mathscr{H}om_{\mathcal{O}_{X,x}}(\mathscr{F}_x,\mathscr{G}_x)\ ,\ (\mathscr{H}om_{\mathcal{O}_{X^h}}(\mathscr{F}^h,\mathscr{G}^h))_x \cong \mathscr{H}om_{\mathcal{O}_{X^h,x}}(\mathscr{F}^h_x,\mathscr{G}^h_x)$$

Put $R = \mathcal{O}_{X,x}$ and $R' = \mathcal{O}_{X^h,x}$. By Lemma 4.1.17 we have

$$(\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F},\mathscr{G})^h)_x = (\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F},\mathscr{G}))_x \otimes_R R'.$$
$$\mathscr{F}^h_x = \mathscr{F}_x \otimes_R R'\ ,\ \mathscr{G}^h_x = \mathscr{G}_x \otimes_R R'$$

By putting all these together we need to prove

$$\mathscr{H}om_R(\mathscr{F}_x,\mathscr{G}_x) \otimes_R R' \cong \mathscr{H}om_{R'}(\mathscr{F}_x \otimes_R R', \mathscr{G}_x \otimes_R R')$$

Since $X = \mathbb{P}^n$, $R = \mathcal{O}_{X,x}$ is a Noetherian ring. By Theorem 4.1.15 $R' = \mathcal{O}_{X^h,x}$ is flat over $R = \mathcal{O}_{X,x}$. Therefore R' becomes a flat R-algebra. In addition, $\mathscr{F}$ is coherent, so $\mathscr{F}_x$ is a finitely generated $\mathcal{O}_{X,x}$-module. Now the above identity is the special case of the lemma 2.2.23. $\square$

Proof of the theorem

Note that we have

$$H^0(X, \mathscr{H}om_{\mathcal{O}_X}(\mathscr{F},\mathscr{G})) = Hom_{\mathcal{O}_X}(\mathscr{F},\mathscr{G}),$$
$$H^0(X^h, \mathscr{H}om_{\mathcal{O}_{X^h}}(\mathscr{F}^h,\mathscr{G}^h)) = Hom_{\mathcal{O}^h_X}(\mathscr{F}^h,\mathscr{G}^h)$$

Consider the morphisms

$$H^0(X, \mathscr{H}om_{\mathcal{O}_X}(\mathscr{F},\mathscr{G})) \xrightarrow{h^0} H^0(X^h, \mathscr{H}om_{\mathcal{O}_X}(\mathscr{F},\mathscr{G})^h) \xrightarrow{\mathrm{i}} H^0(X^h, \mathscr{H}om_{\mathcal{O}_{X^h}}(\mathscr{F}^h,\mathscr{G}^h))$$

Morphism h^0 is an isomorphism because of the first GAGA theorem for $i = 0$. Note that we can apply the theorem since $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F},\mathscr{G})$ is a coherent sheaf by Theorem 2.1.14. Morphism i is an isomorphism because it is the induced isomorphism on global sections from 4.4.2. Therefore $\mathrm{i} \circ h^0$ is an isomorphism completing the proof.

4.5 Proof of theorem 3

4.5.1 Uniqueness

It is followed from the fact that $\mathscr{A}^h$ is fully faithful(GAGA 2). More precisely, Suppose there exist coherent sheaves $\mathscr{F},\mathscr{G}$ such that $\mathscr{F}^h \cong \mathscr{G}^h$. There exist morphisms $f : \mathscr{F}^h \to \mathscr{G}^h, g : \mathscr{G}^h \to \mathscr{F}^h$ such that $f \circ g = id_{\mathscr{G}^h}, g \circ f = id_{\mathscr{F}^h}$. By GAGA theorem 2, there exist morphisms f', g' such that $f'^h = f, g'^h = g$. We have $(f' \circ g')^h = f'^h \circ g'^h = f \circ g = id_{\mathscr{G}^h}$. Since $id^h_{\mathscr{G}} = id_{\mathscr{G}^h}$ and $\mathscr{A}^h$ is faithful, $f' \circ g' = id_{\mathscr{G}}$. Similarly $g' \circ f' = id_{\mathscr{F}}$.

4.5.2 Existence

The proof is by induction on $r = dim\mathbb{P}^r$. The heart of the proof is the following lemma:

Lemma 4.5.1. *Suppose GAGA theorem 3 holds for $r \leq k-1$. Let $X = \mathbb{P}^k_{\mathbb{C}}$(as a projective variety or a projective scheme) and $\mathscr{M}$ be coherent sheaf on X^h. Then there exist an integer N depending only on $\mathscr{M}$ such that for every $x \in X$ and $n \geq N$, $\mathscr{M}_x$ is generated by global sections of $\mathscr{M}(n)$, in other words, the $(\mathcal{O}_{X^h})_x$-module $\mathscr{M}(n)_x$ is generated by elements of $H^0(X^h, \mathscr{M}(n))$.*

Proof of the theorem using the lemma

GAGA 3 is trivial for $k = 0$, suppose the theorem is proved for $r \leq k-1$.
Let $X = \mathbb{P}^k_{\mathbb{C}}$ as a projective scheme or a projective variety, and $\mathscr{M}$ be an arbitrary coherent sheaf on X^h.
By theorem 3.3.1, $H^0(X^h, \mathscr{M}(n))$ is finite dimensional over $\mathbb{C}$, assume $s_1, \ldots, s_m$ are its generators. There exist a morphism $(\mathcal{O}_{X^h})^{\oplus m} \to \mathscr{F}(n)$ by $(a_1, \ldots, a_m) \mapsto a_1 s_1|_U + \ldots + a_m s_m|_U$ for $a_1, \ldots a_m \in \mathscr{M}(n)(U)$ and $U \subset X$ an open set. If n is large enough, the map would be surjective because of lemma 4.5.1. Then we would get a surjective map $(\mathcal{O}_X(-n)^{\oplus m})^h \xrightarrow{i} \mathscr{F}$ by twisting. Let $\mathscr{K}$ the kernel of this map then we get an exact sequence

$$\mathscr{K} \xrightarrow{i} (\mathcal{O}_X(-n)^{\oplus m})^h \to \mathscr{M} \to 0$$

$\mathscr{K}$ is coherent(kernel of a morphism between coherent sheaves is coherent). Since $\mathscr{M}$ was arbitrary, apply the same argument to get a surjective morphism $(\mathcal{O}_X(-n')^{\oplus m'})^h \xrightarrow{j} \mathscr{K}$ for some m', n'. Write $\mathscr{L}_1 = \mathcal{O}_X(-n)^{\oplus m}, \mathscr{L}_2 = \mathcal{O}_X(-n')^{\oplus m'}$. Hence we get an exact sequence

$$\mathscr{L}_2^h \xrightarrow{i \circ j} \mathscr{L}_1^h \to \mathscr{M} \to 0 \tag{8}$$

We have a morphism $\mathscr{L}_2^h \xrightarrow{i \circ j} \mathscr{L}_1^h$, by GAGA 2 theorem, there exist a morphism $\mathscr{L}_2 \xrightarrow{l} \mathscr{L}_1$ such that $l^h = i \circ j$. Take the cokernel of this map to get an exact sequence

$$\mathscr{L}_2 \xrightarrow{l} \mathscr{L}_1 \to \mathscr{F} \to 0$$

Apply the exact functor $-^h$ to get a new exact sequence

$$\mathscr{L}_2^h \xrightarrow{i \circ j} \mathscr{L}_1^h \to \mathscr{F}^h \to 0 \tag{9}$$

Compare (8) and (9) to conclude $\mathscr{F}^h \cong \mathscr{M}$. So we find the desired sheaf $\mathscr{F}$. The theorem is proved for k.

4.5.3 Proof of the lemma

First, we prove a lemma:

Lemma 4.5.2. *Let E be a hyperplane of $\mathbb{P}^k_{\mathbb{C}}$. Let $\mathscr{G}$ be a coherent analytic sheaf on E^h. Then $H^i(E^h, \mathscr{G}(n)) = 0$ for all $i > 0$ and n large enough.*

Proof. Since $dim(E) < K$ by inductive hypothesis there exist an algebraic coherent sheaf $\mathscr{F}$ on E such that $\mathscr{F}^h = \mathscr{G}$. So $\mathscr{G}(n) = \mathscr{F}(n)^h$. By GAGA 1 theorem we have $H^i(E^H, \mathscr{G}(n)) = H^i(E, \mathscr{F}(n))$. By vanishing theorem 2.4.44, $H^i(E, \mathscr{F}(n)) = 0$ for large enough n and $i > 0$.

Proof of the lemma 4.5.1

By hypothesis, GAGA 3 theorem holds for $dimX = r \leq k-1$. Let $x \in X$.

Step 1. We first show that there exist an integet N_x depending on x and $\mathscr{M}$ such that for $n \geq N_x$, $(\mathcal{O}_{X^h})_x$-module $\mathscr{M}(n)_x$ is generated by elements of $H^0(X^h, \mathscr{M}(n))$.

Choose a linear section $t \in \mathcal{O}_X(1)$ vanishing at x, and define E the hyperplane defined by $t = 0$. By 2.4.38 there exsit an exact sequence

$$0 \to \mathcal{O}_{X^h}(-1) \xrightarrow{\times t} \mathcal{O}_{X^h} \to (\mathcal{O}_{X^h})_E \to 0.$$

Tensor with $\mathscr{M}$ to get the exact sequence(tensor product is a right exact functor)

$$0 \to \mathscr{K} \to \mathscr{M}(-1) \to \mathscr{M} \to \mathscr{M}_E \to 0.$$

Where $\mathscr{K}$ is the kernel of $\mathscr{M}(-1) \to \mathscr{M}$. Apply the twisting functor $\mathscr{A}(n)$ we obtain an exact sequence

$$0 \to \mathscr{K}(n) \to \mathscr{M}(n-1) \to \mathscr{M}(n) \to \mathscr{M}_E(n) \to 0.$$

Let $\mathscr{P}$ be the kernel of $\mathscr{M}(n) \to \mathscr{M}_E(n)$. Then we can split the above exact sequence into two short exact sequences

$$0 \to \mathscr{P} \to \mathscr{M}(n) \to \mathscr{M}_E(n) \to 0$$
$$0 \to \mathscr{K}(n) \to \mathscr{M}(n-1) \to \mathscr{P} \to 0$$

By taking cohomology, each sequence gives a long exact sequence

$$H^1(X^h, \mathscr{P}) \to H^1(X^h, \mathscr{M}(n)) \to H^1(X^h, \mathscr{M}_E(n)) \tag{10}$$

$$H^1(X^h, \mathscr{M}(n-1)) \to H^1(X^h, \mathscr{P}) \to H^2(X^h, \mathscr{K}(n)) \tag{11}$$

Clearly $\mathscr{M}_E(n)$ is supported on E. $\mathscr{K}$ is also supported on E since, $\mathscr{K} = Ker(\mathcal{O}_{X^h}(-1) \otimes \mathscr{M} \xrightarrow{\times t} \mathcal{O}_{X^h} \otimes \mathscr{M})$ and E is defined by $t = 0$. Therefore $H^1(X^h, \mathscr{M}_E(n)) = H^1(E^h, \mathscr{M}_E(n)) = 0$ and $H^1(X^h, \mathscr{K}(n)) = H^1(E^h, \mathscr{K}(n)) = 0$ for large enough n, lets say for $n \geq N$, by lemma 4.5.2.

So, for $n \geq N$, the first maps in (10),(11) would be surjective. Hence

$$dim_{\mathbb{C}} H^1(X^h, \mathscr{M}(n-1)) \geq dim_{\mathbb{C}} H^1(X^h, \mathscr{P}) \geq dim_{\mathbb{C}} H^1(X^h, \mathscr{M}(n)) \tag{12}$$

By Theorem 3.3.1 all $H^1(X^h, \mathscr{M}(n)), n \geq 0$ are finite dimensional. So, (12) means that $dim_{\mathbb{C}} H^1(X^h, \mathscr{M}(n))$ is a decreasing sequence for $n \geq N$. Thus there exist $N' > N$ and a constant c such that $dim_{\mathbb{C}} H^1(X^h, \mathscr{M}(n)) = c$ for all $n \geq N'$. Inequlities (5) would be all equality, especially

$$dim_{\mathbb{C}} H^1(X^h, \mathscr{P}) = dim_{\mathbb{C}} H^1(X^h, \mathscr{M}(n)) \tag{13}$$

for all $n > N'$. Since $H^1(X^h, \mathscr{P}) \to H^1(X^h, \mathscr{M}(n))$ is surjective for $n \geq N$, it follows that these maps are isomorphisms as well. Consider again the exact sequence of cohomology

$$H^0(X^h, \mathscr{M}(n)) \xrightarrow{h_1} H^0(X^h, \mathscr{M}_E(n)) \to H^1(X^h, \mathscr{P}) \xrightarrow{h_3} H^1(X^h, \mathscr{M}(n)) \tag{14}$$

Since h_3 is injective and the sequence is exact, h_1 is surjective. What we have achieved so far is that

$$H^0(X^h, \mathscr{M}(n)) \to H^0(X^h, \mathscr{M}_E(n)) \tag{15}$$

is surjectvie for all $n > N'$.

We are going to finish step 1 by Nakayama lemma. Let $R = (\mathcal{O}_{X^h})_x$, $M = \mathscr{M}(n)_x$, $p = (\mathscr{I}_E)_x$. Let N be the R-submodule of M generated by $H^0(X^h, \mathscr{M}(n))$. We should prove $M = N$.

We have $(\mathscr{M}(n))_x = \mathscr{M}(n)_x / (\mathscr{I}_E)_x = M/pM$. Since dim $E <$ dim X, by induction hypothesis, $H^0(X^h, \mathscr{M}_E(n)$ generates M/pM. (15) is a surjection, hence image of N generates M/pM. Therefore, $M = N + pM$. R is a local ring, if $\mathfrak{m}$ is its unique maximal ideal, then $p \subset \mathfrak{m}$. Hence $M = N + \mathfrak{m}M$. Nakayama lemma 2.2.25 gives $M = N$.

Step 2. Now We show that there exist an integer N regardless of x such that for $n \geq N$, any $x \in X$, $(\mathcal{O}_{X^h})_x$-module $\mathscr{M}(n)_x$ is generated by elements of $H^0(X^h, \mathscr{M}(n))$:

Fix a $x \in X$. By part 1, there is a number n_x that $\mathscr{M}(n_x)_x$ is generated by $H^0(X^h, \mathscr{M}(n_x))$. Since $\mathscr{M}(n_x)$ is coherent, there exist an open set $x \in U$ such that $H^0(X^h, \mathscr{M}(n_x))$ generates $\mathscr{M}(n_x)_y$ for $y \in U$.

Lemma 4.5.3. *If $H^0(X^h, \mathscr{M}(n))$ generates $\mathscr{M}(n)_x$, the same property holds for any $\mathscr{M}(m)$ for $m \geq n$.*

Proof. It follows from the fact that $\mathscr{M}(m)$ and $\mathscr{M}(n)$ are locally isomorphic. Let (U_i) be the standard affine open sets of the complex projective space. If $x \in U_k$, then the homomorphism $\mathscr{M}(n) \to \mathscr{M}(m)$, $(f_i)_i \mapsto (f_i(x_k/x_i)^{m-n})_i$ is an isomorphism on U_k, hence an isomorphism on an open neighbourhood of x. □

Because $X^h = P^k_{\mathbb{C}}$(as a complex space) is compact, there are open sets $U_1, \ldots, U_l$ with $n_1, \ldots, n_l$ such that $H^0(X^h, \mathscr{M}(n_i))$ generates $\mathscr{M}(n_i)_x$, $x \in U_i$. Take $n' > max(n_i)$, by the above lemma, $H^0(X^h, \mathscr{M}(n'))$ generates $\mathscr{M}(n')_x$, for all $x \in X$.

4.6 GAGA for projective subscheme/subvariety and proper schemes

In the previous sections, we state and prove GAGA theorems only for projective n-space $\mathbb{P}^n_{\mathbb{C}}$ as a scheme or variety. However, the theorems also hold for projective subvariety and closed subscheme of $\mathbb{P}^n_{\mathbb{C}}$. The proofs are exactly the same but we presented the proofs for only the projective n-space to focus on the main ideas.

Theorem 4.6.1. *Let X be a closed subscheme of $\mathbb{P}^n_{\mathbb{C}}$ or a closed subvariety of projective n-space over $\mathbb{C}$. Let $h : X^h \to X$ the analytification morphism. Then:*

(i) For any coherent sheaf $\mathcal{F}$ on X, the natural morphism

$$H^i(X, \mathcal{F}) \to H^i\left(X^h, \mathcal{F}^h\right)$$

is an isomorphism.

(ii) For any coherent sheaves $\mathcal{F}, \mathcal{G}$ on X, the natural morphism

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}) \to \operatorname{Hom}_{\mathcal{O}_{X^h}}\left(\mathcal{F}^h, \mathcal{G}^h\right)$$

is an isomorphism.

(iii) Every coherent sheaf on X^{an} is isomorphic to $h^*\mathcal{F}$ for a unique coherent sheaf $\mathcal{F}$ on X.

Remark 4.6.2. *If $X = \mathbb{A}^n_{\mathbb{C}}$ then (i) fails, since $H^0(X, \mathcal{O}_X) = \mathbb{C}[z_1, \ldots, z_n]$, $X^h = \mathbb{C}^n$ and there are global holomorphic functions on $\mathbb{C}^n$ which are not polynomials. Therefore we need some sort of compactness. On the other hand, another major difference between affine space and projective space is that the later is proper while the first is not. Grothendieck showed that properness suffices for GAGA theorems.*

Theorem 4.6.3. ***(Grothendieck)*** *Let X be a proper scheme over $\mathbb{C}$, then GAGA theorems hold.*

5 Applications

5.1 Algebraization problem

So far we have discussed a lot about the relationship between the category of coherent sheaves on a space X(the special case of projective scheme) and the category of coherent sheaves on its analytification. But a simple question is:

Algebraization problem : Given a complex analytic space $\mathcal{X}$, does there exist a scheme(or a vareite) X such that $X^h = \mathcal{X}$?

In general the answere is negative. For example, Take $\mathcal{X} = \mathbb{D} \subset \mathbb{C}$ the unit disk. Then there is no scheme X which $X^h = \mathcal{X}$, since for an affine scheme $X = Spec\, \mathbb{C}[X]/(f_1, \ldots, f_k)$ we have $X^h = V(f_1, \ldots, f_k)$ which is a finite set with discrete topology, unless $X^h = \mathbb{C}$.

However, the answere is YES for projective schemes. This is the theorem of chow which we prove it using GAGA theorems.

5.1.1 Chow's theorem and Riemann's existence theorem

Theorem 5.1.1. *(**Chow**) Every closed analytic subspace of the complex space $\mathbb{P}^n$ is the analytification of a subvariety (or a subscheme), in other words every closed analytic subset is Zarisky closed.*

Proof. We prove the variety case. Let $\mathcal{Y} \subset X^h$ be a closed analytic subset. Take the corresponding sheaf $\mathcal{O}_{\mathcal{Y}} = \mathcal{O}_{X^h}/\mathscr{I}$. By Oka coherence theorem, $\mathcal{O}_{\mathcal{Y}}$ is coherent. By GAGA third theorem, there exist a unique coherent sheaf F on X such that $F^h = \mathcal{O}_{\mathcal{Y}}$. Since Zarisky topolology is coarser than usual topology and X, X^h are the same as sets, $Supp\ F = Supp\ F^h = \mathcal{Y}$. $Supp\ F$ is Zarisky closed since it's coherent. Hence $\mathcal{Y}$ is Zarisky closed. □

Remark 5.1.2. *Chow's proof of the theorem in his original 1949 paper was completely an analytic proof.*

Note that the above question for complex manifolds, namely when a complex manifold is algebraic, has a long history. For example, there is a deep theorem due to Riemann that which can be proved with Chow's theorem. It says that all compact Riemannian surfaces are algebraic. This is a surprising theorem, since Riemann surfaces are defined by locally patching charts, then if we add a global condition, namely compactness, the surface is necessarily algebraic!

Corollary 5.1.3. *Every Riemannian surface is an algebraic curve.*

Proof. It follows immediately from the chow's theorem and the following theorem:

Theorem 5.1.4. *A compact Riemann surface can be embedded into the complex projectvie space $\mathbb{P}^n_{\mathbb{C}}$ for some n.*

This theorem can be proved using by two different approaches. The first one uses holomorphic ample line bundle and corollaries of Riemann-Roth theorem and Serre duality. The second one uses modern analysis, namely tools for elliptic PDEs.

□

6 Formal schemes

Formal scheme is a type of space which carries local information about all its infinitesimal surrounding. The importance of local information can be easily seen in complex analysis for analytic functions where all the local information are accessible; an analytic function can be locally represented by a power series. The coefficient of the power series correspond to the local information, namely, all the derivatives of the function. Therefore, when we are given an analytic function, we not only know the value of the function at the specific points but also we know all the local information around each point. This is exactly why by knowing a few things about an analytic function we will know a great deal about its behavior.

In algebraic geometry the local information come from the nilpotent elements in the structure sheaf, and this is where the difference between varieties and schemes becomes important. For example $\{x = 0\}$ and $\{x^2 = 0\}$ defines the same underlying sets but their sheaves, $\mathbb{C}[x]/(x)$ and $\mathbb{C}[x]/(x)^2$ are different. Roughly speaking the sheaf of $\{x^2 = 0\}$ has information about the first derivative. Informally we say that the scheme $Spec\ \mathbb{C}[x]/(x)^2$ is thicker than $Spec\ \mathbb{C}[x]/(x)$.

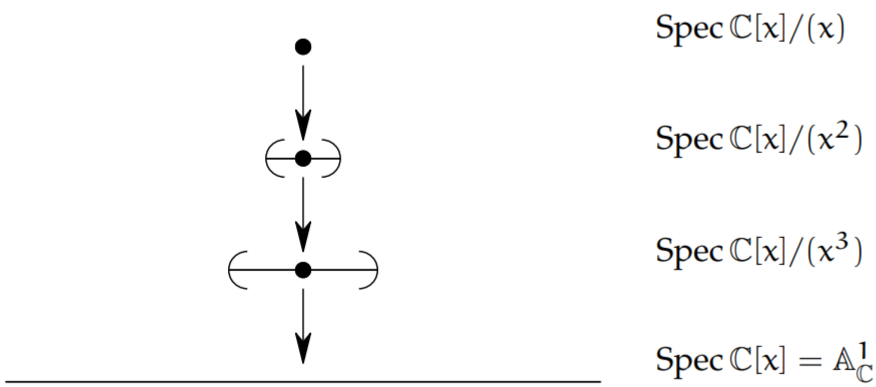

Figure 1: Thickening (Source:[20])

Moreover, if Y is subsceme of an scheme X defined by an ideal sheaf $\mathscr{I}_Y$, we can consider the subschemes Y_n defined by $\mathscr{I}_Y^n$ with nilpotent elements. $(Y_n)_n$ as sets are the same as Y, by their structure sheaves give extra infinitesimal information of Y. Formal completion of Y which we'll define later is an object which carries all the local information in Y_n.

Formal schemes are usually defined for just Noetherian case. We first say a bit about adic noetherian rings and then we define formal scheme and formal completion.

6.1 Adic noetherian ring

Let R be a noetherian ring.

Definition 6.1.1. *A topological ring R is **pre-adic** if there exist an ideal I, called an **ideal of definition**, where the topology of R is the same as the I-adic topology on R. In other words, if the collection of $(I^n)_{n>0}$ is a neighborhood basis for 0.*

Definition 6.1.2. *A pre-adic ring is called **adic** if it is separated and complete.*

Example 6.1.3. *The following are adic rings:*

(i) $R[[x_1, \ldots, x_n]]$ eqipped with the I-adic topology for $I = (x_1, \ldots, x_n)$. Simply it is the completion of the ring $R[x_1, \ldots, x_n]$.

(ii) The ring of integers of a non-archimedean field.

Lemma 6.1.4. *Let R be a pre-adic ring and I, J two ideal of defintion. Then there exist positive numbers m, n such that $J^n \subset I$ and $I^m \subset J$.*

Proof. It immediately follows from the fact that $(I^n)_{n>0}$ and $(J^n)_{n>0}$ are neighborhood basis for 0. $\square$

Lemma 6.1.5. *Let $\mathfrak{p}$ be a prime ideal of a pre-adic ring R. The following are equivalent:*

(i) $\mathfrak{p}$ contanins as ideal of defintion.

(ii) $\mathfrak{p}$ contanins every the ideal of defintion.

(iii) $\mathfrak{p}$ is open.

Proof. If $I \subset \mathfrak{p}$, by the above lemma $J^n \subset \mathfrak{p}$. Since $\mathfrak{p}$ is prime $J \subset \mathfrak{p}$. Suppose $I \subset \mathfrak{p}$ for an ideal of definion I. If $x \in \mathfrak{p}$, $x + I \subset \mathfrak{p}$ is an open neighborhood of x. $\square$

Lemma 6.1.6. *If I is an ideal of definition, so are I^n for $n > 0$.*

Corollary 6.1.7. *Let R be an pre-adic ring. The set $|Spec\ R/I^m|$ is the same for all ideal of defintion I an all $m > 0$. In fact $|Spec\ R/I^m|$ consists of open prime ideals of R.*

Lemma 6.1.8. *Let R be a pre-adic ring. The following are equivalent:*

(i) I is the largest ideal of definition.

(2) R/I is a reduced ring.

(3) I is the maximal ideal of definition.

Proof. Let $J \not\subset I$ another ideal of defintion. For some m we have $J^m \subset I$. If $x \in I/J$ then $(x + I)$ is nonzero nilpotent element of R/I. In addition, if $0 \neq [x] \in R/I$ is nilpotent, $I + (x)$ is a larger ideal of definition than I.

$\square$

Corollary 6.1.9. *If R is Noetherian it has a largest ideal of definition.*

Remark 6.1.10. *Much of the theory of formal scheme is based on the fact that there exist such a largest ideal of definition, this is why we assume R is Noetherian in the beginning.*

6.2 Formal spectrum

As with schemes which constructed using prime spectrum of a commutative ring, we will construct *formal schemes* using *formal spectrum* of an adic noetherian ring.

Definition 6.2.1. *Fix an adic noetherian ring (R, I) with I an ideal of definition. Define affine schemes $(R_n, \mathcal{O}_{R_n}) = Spec(R/I^{n+1})$. Note that each R_n has the same underlying space $|R_n| =$ {prime ideals of R/I}. Natural projections $I^{n+1} \to I^{m+1}$ for $n \geq m$ gives scheme morphisms $R_m \to R_n$ for $n \geq m$. These give us an inverse system of ringed space. Taking inverse limit in the category of ringed space we obtain a ringed space called **formal spectorm of** R:*

$$(\mathcal{X} = \operatorname{Spf} R, \mathcal{O}_{\mathcal{X}}) = \varprojlim R_n$$

where $|\mathcal{X}| = |R_0|$ and $\mathcal{O}_{\mathcal{X}} = \varprojlim \mathcal{O}_{R_n}$.

Theorem 6.2.2. $\operatorname{Spf} R$ *only depends on R as a topological ring. It doesn't change if we replace I by another ideal of definition. In fact,* $\operatorname{Spf} R$ *is the subspace of $Spec(R)$ consisting of open prime ideals, and*

$$\operatorname{Spf} R = \varprojlim_{\lambda} Spec\ (R/I_\lambda)$$

when (I_λ) are the ideals of definition of R.

Proof. If J is another ideal of definition, there are m, n with $J^n \subset I$ and $I^m \subset J$. Hence for each k we have the morphisms $Spec\ R/I^k \to Spec\ R/J^{nk}$ and $Spec\ R/J^k \to Spec\ R/I^{mk}$. Therefore $(SpecR/I^k)_k$ and $(SpecR/J^k)_k$ are cofinal with each other and so have the same inverse limit. The fact about open prime ideals follows from corollary 6.1.7.

□

Lemma 6.2.3. ***Some properties of*** $\operatorname{Spf} R$ ***:***

1. *we have $\mathcal{O}_{\mathcal{X}}(U) = \varprojlim \Gamma(U, \mathcal{O}_{R_n})$ for any open subset $U \subseteq \operatorname{Spf} R$. In particular $\mathcal{O}_{\mathcal{X}}(\mathcal{X}) = \varprojlim R/I^n = \widehat{R}_I = R$, since R is complete by the definition.*

2. $\operatorname{Spf} R$ *consists of all open primes of R.*

3. *For each $f \in R$, Define $D_f = \{$ open $\mathfrak{p} \in \operatorname{Spec} R : f \notin \mathfrak{p}\}$. By defintion we would have*

$$\mathcal{O}_{\mathcal{X}}\left(D_f\right) = \varprojlim R_f/I^{n+1}R_f = \widehat{R_f}$$

where $\widehat{R_f}$ is the I-adic completion of R_f with respect to IR_f.

4. Let $x \in X$ be a point defined by an open ideal $\mathfrak{p} \in \operatorname{Spec} R$. We have $\mathcal{O}_{\mathcal{X},x} = \varprojlim_{f(x)\neq 0} \widehat{R_f}$. By noetherian condition we can prove it is a local ring with the maximal ideal $\mathfrak{p}\mathcal{O}_{\mathcal{X},x}$.

Lemma 6.2.4. *Every continuous ring homomorphism of $A \to B$ of adic rings induces a morphism of formal spectrum $\operatorname{Spf} B \to \operatorname{Spf} A$.*

6.3 Locally Noetherian formal schemes

Definition 6.3.1. *A **topologically ringed space** $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a ringed space whose sheaf of rings is a sheaf of topological rings.*

Definition 6.3.2. *An **affine formal scheme** $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a topologically ringed space isomorphic to he formal spectrum $SpfA$ for some adic roethreian ring A.*

Definition 6.3.3. *A **locally noetherian formal scheme** $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a topologically locally ringed space that is locally an affine formal scheme. It means for any $x \in \mathcal{X}$ there is an open neighborhoods $x \in U$ such that $(U, \mathcal{O}_{\mathcal{X}}|_U)$ is an affine formal scheme.*

A morphism $f : \mathcal{X} \to \mathcal{Y}$ between locally noetherian formal scheme is a morphism of locally ringed spaces such that it is *continous*, meaning that four every open set $V \subset Y$, the induced map $f : \Gamma(V, \mathcal{O}_{\mathcal{Y}}) \to \Gamma(f^{-1}(V), \mathcal{O}_{\mathcal{X}})$ is a continuous map between topological rings.

locally noetherian formal scheme form a category with above morphisms.

Like ordinary schemes, it is easy to see that $Mor(SpfA, SpfB) = Hom_{cont}(B, A)$. Where Hom_{cont} means continuous maps between two topological rings.

6.4 Ideals of definition and Coherent sheaves

In the previous sections, we saw that a sheaf on a noetherian scheme is coherent, if on local affine sets Spec R, it is isomorphic to $\tilde{M}$ for a finitely generated R-module M. Similar result holds for coherent sheaves on Locally Noetherian formal schemes.

Definition 6.4.1. *Let R be an adic ring. For an R-module M define*

$$M^{\triangle} = \varprojlim_{\lambda} (\widetilde{M/I_\lambda M})$$

where I_λ are ideals of definition of R. Clearly $M^{\triangle}$ is a sheaf of $\mathcal{O}_R$-module on $Spf\ R$.

Remark 6.4.2. *If $I \subset R$ is an ideal of defintion, then $M^{\triangle} = \varprojlim_{n} (\widetilde{M/I^{n+1}M})$*

Lemma 6.4.3. *$\mathscr{F}$ is cohrent on a Locally Noetherian formal scheme $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ If and only if for any open affine formal scheme $U = \operatorname{Spf} R$, we have $\mathscr{F}|_U \cong M^{\triangle}$ for a finitly generated R-module M.*

As a result, $\mathcal{O}_{\mathcal{X}}$ is a coherent sheaf of ring.

Definition 6.4.4. *An* ***ideal of definition of a formal scheme*** $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ *is a coherent ideal sheaf* $\mathcal{I} \subset \mathcal{O}_{\mathcal{X}}$ *such that for any point* $x \in \mathcal{X}$*, there is an open affine neighbourhood* $U =$*Spf R such that* $\mathcal{I}|_U \cong I^{\triangle}$ *for some ideal of definition I of R.*

Theorem 6.4.5. *A coherent ideal sheaf* $\mathcal{I} \subset \mathcal{O}_{\mathcal{X}}$ *is an ideal of definition if and only if* $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}/\mathcal{I})$ *is a scheme.*

Proof. Supose $\mathcal{I}$ is an ideal of definition. Choose an open formal affine set $U = Spf\ R$ of $\mathcal{X}$ with $\mathcal{I}|_U \cong I^{\triangle}$. We prove $(U, (\mathcal{O}_{\mathcal{X}}/\mathcal{I})|_U) \cong Spec\ R/I$. $(\mathcal{O}_{\mathcal{X}}/\mathcal{I})|_U)$ is the sheafification of the presheaf $V \mapsto \mathcal{O}_{\mathcal{X}}|_U(V)/\mathcal{I}|_U(V)$. Choose $f \in R$ with $D_f \subset U$. By lemma 6.2.3 we have $\mathcal{O}_{\mathcal{X}}|_U(D_f)/\mathcal{I}|_U(D_f) \cong \widehat{R_f}/\widehat{I_f} \cong R_f/I_f \cong (R/I)_f = \mathcal{O}_{Spec\ R/I}(D_f)$. So $(U, (\mathcal{O}_{\mathcal{X}}/\mathcal{I})|_U)$ and $Spec\ R/I$ agree locally on open basis D_f, hence equal.

□

Note that the schemes $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}/\mathcal{I})$ have the same underlying space as formal scheme $\mathcal{X}$.

Theorem 6.4.6. *Let* $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ *be a locally noetherian formal scheme. There exist a unique ideal of definitions* $\mathcal{J}_{\mathcal{X}}$ *such that* $(\mathcal{X}, \mathcal{O}_{\mathcal{X}}/\mathcal{J}_{\mathcal{X}})$ *is a reduced scheme. Morever, a coherent ideal sheaf* $\mathcal{I} \subset \mathcal{O}_{\mathcal{X}}$ *is an ideal of definition if and only if* $\mathcal{J}_{\mathcal{X}}^n \subset \mathcal{I} \subset \mathcal{J}_{\mathcal{X}}$ *for some* $n > 0$*. This means* $\mathcal{J}_{\mathcal{X}}$ *is the largest ideal of definition.*

Proof. See [6] proposition 9.5 . □

Remark 6.4.7. *From the above points, it is clear that if* $\mathcal{I}$ *is an ideal of definition, so is* $\mathcal{I}^m$ *for* $m \geq 1$.

Lemma 6.4.8. *Let R be an adic ring and* $I \subset R$ *an ideal of definition. Let* $X = Spec\ R$ *and* $\mathcal{X} = Spf\ R$*. Then:*

(i) For the sheaf of ideal $\mathcal{I} = I^{\triangle}$ *we have* $\mathcal{O}_{\mathcal{X}}/\mathcal{I}^n \cong \widetilde{(R/I^n)}$

(ii) If M is a finitely generated R-module, then $M^{\triangle} = \tilde{M} \otimes_{\mathcal{O}_X} \mathcal{O}_{\mathcal{X}}$.

(iii) The functor $M \mapsto M^{\triangle}$ *is an exact functor from the category of R-modules to the category of coherent* $\mathcal{O}_{\mathcal{X}}$*-modules.*

Proof. See [6] proposition 9.4. □

Theorem 6.4.9. *Let R be an adic ring and* $I \subset R$ *an ideal of definition.* $\mathcal{X} = Spf\ R$*. Then there is an equivalence of categories between the category of finitely generated R-modules and the category of coherent* $\mathcal{O}_{\mathcal{X}}$*-modules by exact functors:*

$$M \mapsto M^{\triangle} \text{ and } \mathscr{F} \mapsto \mathscr{F}(\mathcal{X})$$

In particular, every coherent $\mathcal{O}_{\mathcal{X}}$*-module* $\mathscr{F}$ *is of the form* $M^{\triangle}$ *for some finitely generated R-module M.*

6.5 Thickening

Fix an ideal of definition $\mathcal{I}$ of $\mathcal{X}$. We have the locally noetherian schemes $X_n = (\mathcal{X}, \mathcal{O}_{\mathcal{X}}/\mathcal{I}^{n+1})$ which form a sequence

$$X_\bullet = X_0 \to X_1 \to X_2 \dots$$

Theorem 6.5.1. *We have $|X_n| = \mathcal{X}$ and $\mathcal{O}_{\mathcal{X}} = \varprojlim_n \mathcal{O}_{\mathcal{X}_n}$. That means $\mathcal{X}$ can be described as the colimit of nilpotent thickening as schemes.*

Example 6.5.2. *Let R be an adic ring. Let $\mathcal{X} = Spf\ R$ and $\mathcal{I} = I^{\triangle}$ for some ideal of defintion $I \subset R$. Then $X_n = Spec\ (R/I^{n+1})$.*

Conversely we have

Lemma 6.5.3. *Consider a sequence of ringed space*

$$X_\bullet = X_0 \to X_1 \to X_2 \dots$$

satisfying:

(i) X_0 is a locally noetherian scheme.

(ii) The underlying topological space of all X_i are homeomorphic and with this identification the morphisms $\mathcal{O}_{X_{n+1}} \to \mathcal{O}_{X_n}$ are surjective.

(iii) Let J_n=Ker $\mathcal{O}_{X_n} \to \mathcal{O}_{X_0}$, then for $m \leq n$, Ker $\mathcal{O}_{X_n} \to \mathcal{O}_{X_m} = J_n^{m+1}$

(iv) J_1 is coherent $\mathcal{O}_{X_0}$-module.

Then $(\mathcal{X}, \varprojlim_n \mathcal{O}_{X_n})$ is a locally noetherian formal scheme.

Let $\mathcal{X}$ be a formal scheme and $\mathcal{I}$ an ideal of definition. Let $X_n = (\mathcal{X}, \mathcal{O}_{\mathcal{X}}/\mathcal{I}^{n+1})$ the locally noetherian schemes. Let $u_n : X_n \to \mathcal{X}$ be the natural morphism of ringed spaces. For any $\mathcal{O}_{\mathcal{X}}$-module $\mathscr{F}$, we have the sheaves of $\mathcal{X}_n$-modules $\mathscr{F}_n = u_n^* \mathscr{F}$ on X_n.

Lemma 6.5.4. *If $\mathscr{F}$ is a coherent sheaf of $\mathcal{O}_{\mathcal{X}}$-modules, then each $\mathscr{F}_n$ is a coherent sheaf of $\mathcal{O}_{X_n}$-modules, and $\mathscr{F} = \varprojlim_n \mathscr{F}_n$.*

Proof. Coherence is a local notion, assume $\mathcal{X} = Spf\ R$ is affine and $\mathscr{F} \cong M^{\triangle}$ for a finitely generated R-module M. Then $\mathscr{F}_n = \widetilde{M/I^n M}$ which is coherent on $X_n = Spec\ (R/I^n)$. $\square$

Let $f : \mathcal{X} \to \mathcal{Y}$ be a morphism of locally noetherian formal schemes, and $\mathcal{J}$ an ideal of definition of $\mathcal{Y}$. Since f is continous, $f^*(\mathcal{J})\mathcal{O}_{\mathcal{X}} \subset \mathcal{J}_{\mathcal{X}}$ (because $f^\sharp : f^*\mathcal{O}_{\mathcal{Y}} \to \mathcal{O}_{\mathcal{X}}$ should be a morphism of toplogical ringed space).

Fix an ideal of defintion $\mathcal{I}$ such that $f^*(\mathcal{J})\mathcal{O}_{\mathcal{X}} \subset \mathcal{I}$. We have $f^*(\mathcal{J}^{n+1})\mathcal{O}_{\mathcal{X}} \subset \mathcal{I}^{n+1}$. Therefore , f induces morphisms of $f_n : X_n \to Y_n$, which gives rise to a morphism of sequences $f : X_\bullet \to Y_\bullet$ and the following diagram

$$\begin{array}{ccc} X_n & \xrightarrow{u_n} & \mathcal{X} \\ \downarrow f_n & & \downarrow f \\ Y_n & \xrightarrow{u_n'} & \mathcal{Y} \end{array}$$

commutes. f is the colimit of the morphisms f_n that makes the above squares commutative.

Definition 6.5.5. *Usually, $f^*(\mathcal{J})\mathcal{O}_{\mathcal{X}}$ is not an ideal of definition, if it is, we call f an **adic morphism.**. In this case we call $\mathcal{X}$ a **$\mathcal{Y}$-adic formal scheme**.*

Lemma 6.5.6. *f is an adic morphism if and only if the above diagram is cartesian.???*

6.6 Formal completion along a closed subscheme

In this section we want to do the opposite of what we did in the previous section. In before, we got locally Noetherian schemes X_n from a locally Noethereian formal scheme $\mathcal{X}$. Now we want to make a formal scheme from a locally Noetherian scheme. This construction is similar to the completion of a ring with respect to an ideal.

Let $(X, \mathcal{O}_X)$ be a locally noetherian scheme, and $X' \subset X$ be a closed subscheme defined by an ideal of sheaf $\mathscr{I} \subset \mathcal{O}_X$. Define $X_i = (X', \mathcal{O}_X/\mathscr{I}^{n+1})$. Note that all X_i have the same underlying space, namely X'. Morphisms $\mathcal{O}_X/\mathscr{I}^n \to \mathcal{O}_X/\mathscr{I}^m$ for $n > m$ define a sequence

$$X_\bullet = X_0 \to X_1 \to X_2 \ldots$$

that satisfies all the condition of lemma 6.5.3. Therefore the colimit $\hat{X} = \varinjlim_n X_n$ is a formal scheme, namely the completion of X along X'.

Example 6.6.1. *Let $X = SpecR$ be an affine scheme, and $X' = SpecR/I$, then $\hat{X} = Spf\hat{R}$, where $\hat{R}$ is the I-adic completion of R.*

Remark 6.6.2. *If X is a locally Noetherian scheme, take $X' = X$. Then $\hat{X} = X$. Therefore, the category of locally noetherian formal schemes contains all Noetherian schemes.*

Lemma 6.6.3. *Let X be a locally noetherian scheme, and $X' \subset X$ be a closed subscheme. If $\mathscr{I}$ and $\mathscr{J}$ are ideal sheaves defining X', then:*

(i) there are integers m, n such that $\mathscr{J}^n \subset \mathscr{I}$ and $\mathscr{I}^m \subset \mathscr{J}$.

(ii) the inverse system $(\mathcal{O}_X/\mathscr{I}^{n+1})$ and $(\mathcal{O}_X/\mathscr{J}^{n+1})$ are cofinal with each other, hence have the same inverse limit.

Corollary 6.6.4. *$\hat{X}$ does not depend on the choice of the ideal sheaf I. In fact $|\hat{X}| = |X'|$ and $\mathcal{O}_{\hat{X}} = \varprojlim_{\mathscr{I}} \mathcal{O}_X/\mathscr{I}$ where $\mathscr{I}$ runs through all coherent ideal sheaf of $\mathcal{O}_X$ such that Supp $\mathcal{O}_X/\mathscr{I} = X'$.*

The immersions $i_n : X_n \to X$ define a morphism of ringed spaces

$$i : \hat{X} \to X$$

There is a commutative diagram

$$\begin{tikzcd} & \hat{X} \arrow[dr, "i"] & \\ X_n \arrow[ur, "u_n"] \arrow[rr, "i_n"] & & X \end{tikzcd}$$

For an $\mathcal{O}_X$−module F define $F_n = i_n^* F = F/\mathscr{I}^{n+1}F$ which is an $\mathcal{O}_{X_n}$-module. It is an inverse system of sheaves, by taking limit one gets an $\mathcal{O}_{\hat{X}}$−module $\hat{F} = \varprojlim_n F_n$, which is called the **completion of** F

Lemma 6.6.5. *(1) i is flat.*

(2) If F is coherent then $i^*F = \hat{F}$.

Theorem 6.6.6. *Let X be a locally noetherian scheme, and $X' \subset X$ be a closed subscheme. Let $\mathcal{X} = \hat{X}$ the completion along X'. Then:*

(i) the functor $\mathscr{F} \mapsto \hat{\mathscr{F}}$ is an exact functor from the category of coherent $\mathcal{O}_X$-modules to the category of coherent $\mathcal{O}_{\mathcal{X}}$-modules.

(ii) If $\mathscr{I}$ is the sheaf of ideals defining X', then $\hat{\mathscr{F}}/\hat{\mathscr{I}}^n\hat{\mathscr{F}} \cong \mathscr{F}/\mathscr{I}^n\mathscr{F}$.

(iii) $\hat{\mathscr{F}} \cong \mathscr{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{\mathcal{X}}$.

Proof. Since all the claims are local the theorem reduces to the affine case which is stated in lemma 6.4.8 and 6.4.9. ☐

Let $f : X \to Y$ be a morphism of locally noehterian schemes, and X', Y' closed subscheme of X, Y such that $f(X') \subset Y'$. Let $\hat{X}, \hat{Y}$ be the completion along X', Y' respectively. We have a commutative diagram

$$\begin{tikzcd} X_n \arrow[r, "f_n"] \arrow[d] & Y_n \arrow[d, "f"] \\ X_m \arrow[r, "f_m"] & Y_m \end{tikzcd}$$

for $n \geq m$. Therefore by taking limit, f defines a morphism $\hat{f} : \hat{X} \to \hat{Y}$. $\hat{f}$ is addic if and only if $f^{-1}(Y') = X'$.

6.7 Closed formal subscheme

Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a locally noetherian formal scheme.

Definition 6.7.1. *Let $\mathcal{A}$ be a coherent ideal of $\mathcal{O}_{\mathcal{X}}$. Let $\mathcal{Y} = Supp\ \mathcal{O}_{\mathcal{X}}/\mathcal{A}$ which is a topologically ringed space. We call $(\mathcal{Y}, \mathcal{O}_{\mathcal{X}}/\mathcal{A}_{\mathcal{Y}})$ the **closed formal subscheme** of $\mathcal{X}$ defined by $\mathcal{A}$.*

Lemma 6.7.2. *(1) $i : \mathcal{Y} \to \mathcal{X}$ is adic.*

(2) If $\mathcal{I}$ is an ideal of defintion of $\mathcal{X}$, $X_n = (\mathcal{X}, \mathcal{O}_\mathcal{X}/\mathcal{I}^{n+1}), \mathcal{X} = \varinjlim X_n$, then $\mathcal{Y} = \varinjlim Y_n$ where Y_n is a closed subscheme of X_n with $\mathcal{O}_{Y_n} = \mathcal{O}_{X_n} \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{O}_Y$.

(3) if X is a locally noetherian scheme and $\hat{X}$ is its completion along a closed subscheme X_0, if Y is a closed subscheme of X and $\hat{Y}$ its completion along $Y_0 = X_0 \times_X Y$, then $\hat{Y}$ is a closed formal subscheme of X.

6.8 Finite maps

Let $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ be a locally noetherian formal scheme. Fix an ideal of defintion $\mathcal{I} \subset \mathcal{X}$, $X_n = (\mathcal{X}, \mathcal{O}_\mathcal{X}/\mathcal{I}^{n+1}), \mathcal{X} = \varprojlim X_n$.

Definition 6.8.1. *Let f be an adic morphism $f : \mathcal{Z} \to \mathcal{X}$ of locally noetherian formal schemes. Let $Z_n = X_n \times_\mathcal{X} \mathcal{Z}$. f is called **finite** if $f_0 : Z_0 \to X_0$ is a finite map of schemes. It is equivalent to saying each map $f_n = Z_n \to X_n$ is finite.*

Lemma 6.8.2. *If X is a locally noetherian scheme and $Z \to X$ a finite X-scheme, and $\hat{X}$ is its completion along a closed subscheme X_0, then the completion $\hat{Z}$ of Z along $Z_0 = X_0 \times_X Z$ is a finite map over $\hat{X}$.*

Proof. This is because base change of finite map is finite. Therefore, since $Z \to X$ is finite, $Z_0 \to X_0$ is also finite. □

7 Formal GAGA

In this section we want to present the Formal GAGA theorems. They are analogous theorems of GAGA which hold for formal schemes where the analytification process is the completion along a closed subscheme. The theorems are the consequences of two major theorems: The comparison theorem and the Existence theorem. They were first proved by Grothendieck. First we need some background. That would be some familiarity with the higher direct image functors and base change maps. Later on for the proof of the theorems we need more technical tools such as Mittag-Leffler conditions.

7.1 Preliminaries

7.1.1 Chow's lemma

Chow's lemma is an important results in algebraic geometry proved by Wei-Liang Chow. It has several versions, we need the Noetherian version in the proof of existence theorem.

Lemma 7.1.1. ***(Chow's lemma)***
Let Y be a noetherian scheme. Let $f : X \to Y$ be a separated morphism of finite type. Then there exist morphisms

$$Z \xrightarrow{g} X \xrightarrow{f} Y \tag{16}$$

where g is projective (hence proper) and surjective and fg is quasi-projective. Moreover, there exist an open $U \subset X$ where induced map $g : g^{-1}(U) \to U$ is an isomorphism.

Proof. See [4]. □

7.1.2 Higher Direct Images of Sheaves

Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Let $Mod(X)$ be the abelian category of $\mathcal{O}_X$-modules on X. Define $Mod(Y)$ similarly.

Definition 7.1.2. *We have the direct image functor $f_* : Mod(X) \to Mod(Y)$. This is a left exact functor. Since $Mod(X)$ and $Mod(Y)$ have enough injectives, define the derived functors of f_* by $R^i f_* : Mod(X) \to Mod(Y)$ for $i \geq 0$, called the **higher direct image** sheaves.*

By the definition of the derived functors, we immediately get

Lemma 7.1.3. *(1) $R^0 f_*(\mathscr{F}) \cong f_*(\mathscr{F})$ for all $\mathcal{O}_X$-module $\mathscr{F}$.*

(2) For a short exact sequence of sheaves

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$$

There are connecting homorphisms $R^i f_*(\mathscr{F}'') \to R^{i+1} f_*(\mathscr{F}')$ and a long exact sequence of $\mathcal{O}_Y$-module

$$0 \to f_*(\mathscr{F}') \to f_*(\mathscr{F}) \to f_*(\mathscr{F}'') \to R^1 f_*(\mathscr{F}') \to R^1 f_*(\mathscr{F}) \to \dots$$

Lemma 7.1.4. *Let $f : X \to Y$ be a morphism of ringed space, and $g : Y \to Z$ a closed embedding. If $\mathscr{F}$ is a sheaf of modules on X, then there is a canonical isomorphisms*

$$g_* R^i f_*(\mathscr{F}) \to R^i(gf)_*(\mathscr{F})$$

for all $i \geq 0$.

The higher direct image sheaves relate to the cohomology groups as follows:

Theorem 7.1.5. *Let $f : X \to Y$ be a morphism of ringed space, and $\mathscr{F}$ be a sheaf of modules on X. Then $R^i f_*(\mathscr{F})$ is the sheafification of the presheaf*

$$U \to H^i(f^{-1}(U), \mathscr{F})$$

7.1.3 Base change maps

Consider a commutative square of ringed spaces

$$\begin{array}{ccc} X' & \xrightarrow{h} & X \\ \big\downarrow{\scriptstyle f'} & & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow{g} & Y \end{array}$$

Lemma 7.1.6. *Let F be an $\mathcal{O}_X$ module. Then there is a canonical map of $\mathcal{O}_{Y'}$-modules*

$$\lambda : g^* f_* F \to f'_* h^* F \tag{17}$$

*called the **base change map**.*

λ can be defined in two equivalent ways:

(i) We have the adjunction map $F \to h_* h^* F$. Apply f_* to get

$$\lambda_1 : f_* F \to f_* h_* h^* F = g_* {f'}_* h^* F$$

Adjunction of g_*, g^* gives (17).

(ii) We have the adjunction map $f^* f_* F \to F$. Apply h^* to get

$$\lambda_2 : h^* f^* f_* F = {f'}^* g^* f_* F \to h^* F$$

Adjunction of ${f'}_*, {f'}^*$ gives (17).

Remark 7.1.7. *The fact that (i) and (ii) are equivalent is not easy and was proved in great generality by P. Deligne.*

In general, there are base change maps by higher direct images of sheaves:

Lemma 7.1.8. *There are canonical maps of $\mathcal{O}_{Y'}$-modules*

$$\lambda : g^* R^q f_* F \to R^q f'_*(h^* F) \tag{18}$$

*for all $q \geq 0$, also called the **base change maps**.*

By adjunction between g^*, g_* it suffice to have a map

$$\lambda_1 : R^q f_* F \to g_* R^q f'_*(h^* F)$$

λ_1 is the composition of two maps α and β:

$$\alpha : R^q f_* F \to R^q a_*(h^* F) \tag{19}$$

$$\beta : R^q a_*(h^* F) \to g_* R^q f'_*(h^* F) \tag{20}$$

where $a = fh = gf'$.

The map α **:** Note that $R^q f_* F$ is the sheaf associated to the presheaf $V \to H^q(f^{-1}V, F)$ for open set $V \subset Y$. $R^q a_*(h^* F)$ is the sheaf associated to the presheaf $V \to H^q(a^{-1}V, h^* F)$. Now α is the functorial cohomology map $H^q(f^{-1}V, F) \to H^q(a^{-1}V, h^* F)$ corresponding to the induced map $h : a^{-1}V \to f^{-1}V$.

The map β **:** By definition, $g_* R^q f'_*(h^* F)(V) = H^0(g^{-1}(V), R^q f'_*(h^* F))$. β is the restriction map

$$H^q(a^{-1}V, h^* F) \to H^0(g^{-1}(V), R^q f'_*(h^* F))$$

7.1.4 The finiteness theorem for proper morphisms

Theorem 7.1.9. *Let $f : X \to Y$ be a proper morphism of schemes. If Y is locally noetherian, and $\mathscr{F}$ is a coherent sheaf on X, then the sheaves $R^i f_*(\mathscr{F})$ are coherent for all $i \geq 0$.*

Another variant of the theorem we need is:

Theorem 7.1.10. *Let $f : X \to Y$ be a proper morphism, with Y noetherian. Let $S = \bigoplus_{n\in\mathbb{N}} S_n$ be a quasi-coherent, graded $\mathcal{O}_Y$-algebra of finite type over S_0 and generated by S_1. Let $M = \bigoplus_{n\in\mathbb{Z}} M_n$ be a quasi-coherent, graded sheaf $f^*(S)$-module of finite type. Then, for all $q \in \mathbb{Z}$,*

$$R^q f_* M := \bigoplus_{n\in\mathbb{Z}} R^q f_* M_n$$

is a graded S-module of finite type, and there exists an integer n_0 such that, for any $n \geq n_0$

$$R^q f_* M_n = S_{n-n_0} R^q f_* M_{n_0}$$

Corollary 7.1.11. *Let R be a noetherian ring and X be a noetherian scheme. Suppose $f : X \to Spec\ R$ is a finite type morphism. Let $I \subset R$ be an ideal and $f^*(\hat{I}) = \mathcal{I}$. Let F be a coherent sheaf on X whose support is proper over $Spec\ R$. I can be considered as the subring of $\mathcal{O}_X(X)$ and therefore can act on $H^q(X.\mathscr{G})$ for $\mathcal{O}_X$-modules $\mathscr{G}$. Let $B = \bigoplus_{n\in\mathbb{N}} I^n$.*

Then for all q, $\bigoplus_{n\in\mathbb{N}} H^q(X, \mathcal{I}^n F)$ is a finitely generated graded B-module, and there exist n_0 such that, for any $n \geq n_0$ $H^q(X, I^n F) = I^{n-n_0} H^q(X, I^{n_0} F)$.

Proof. Since $B = \bigoplus_{n\in\mathbb{N}} I^n$ is a finitely generated R-algebra, it can be considered as a quasi-coherent sheaf S of finitly generated $\mathcal{O}_{Spec\ R}$ algebras on $Spec\ R$. If we pullback it to X we get an $\mathcal{O}_X$-algebra $f^*(S) = \bigoplus_{n\in\mathbb{N}} \mathcal{I}^n$. Now the quasi-coherent sheaf $M = \bigoplus_{n\in\mathbb{Z}} \mathcal{I}^n \mathscr{F}$ is a finitely generated graded sheaf of modules over $f^*(S)$. Then by the above theorem and by taking global sections we get that $\bigoplus_{n\in\mathbb{N}} H^q(X, \mathcal{I}^n F)$ is a finitely generated graded B-module and there exist n_0 satisfying the corollary. $\square$

There is another nontrivial theorem we need about finiteness of cohomology groups.

Theorem 7.1.12. *Let R be a noetherian ring and $f : X \to Spec\ R$ a proper morphism. Then all cohomology groups of any coherent sheaf on X are finitely generated R module.*

7.2 Completion morphism

Let $f : X \to Y$ be a morphism of locally noetherian schemes, $Y \subset Y'$, $X' = f^{-1}(Y')$. Let $\hat{X}, \hat{Y}$ formal completion along X', Y'. We have the commutative diagram,

$$\begin{tikzcd} \hat{X} \arrow[r, "i"] \arrow[d, "\hat{f}"] & X \arrow[d, "f"] \\ \hat{Y} \arrow[r, "i"] & Y \end{tikzcd}$$

Let $\mathscr{F}$ be a sheaf of $\mathcal{O}_X$-module. By lemma 7.1.8 there are base change maps of $\mathcal{O}_{\hat{Y}}$-modules

$$i^* R^q f_* \mathscr{F} \to R^q \hat{f}_* (i^* \mathscr{F})$$

We want to simplify the above map. If $\mathscr{F}$ is coherent, by lemma 6.6.5, $i^* \mathscr{F} = \hat{\mathscr{F}}$.

Lemma 7.2.1. *If one of the following holds, then $R^q f_* \mathscr{F}$ is coherent.*

(1) f is proper.

(2) f is of finite type and the support of $\mathscr{F}$ is proper over Y.

Therefor, add one of the above condition and suppose $\mathscr{F}$ is coherent. Then again by previous lemmas, $i^* R^q f_* \mathscr{F} = \widehat{R^q f_* \mathscr{F}}$. Hence, the base change maps become

$$\widehat{R^q f_* \mathscr{F}} \to R^q \hat{f}_* (\hat{\mathscr{F}}) \tag{21}$$

In addition, from the commutative diagram

$$\begin{tikzcd} X_n \arrow[r, "u_n"] \arrow[d, "f_n"] & \hat{X} \arrow[d, "\hat{f}"] \\ Y_n \arrow[r, "v_n"] & \hat{Y} \end{tikzcd}$$

we get the base change maps

$$v_n^* R^q \hat{f}_* \hat{\mathscr{F}} \to R^q f_{n*} ((u_n^* \hat{\mathscr{F}}))$$

By adjunction, we get the morphisms

$$R^q \hat{f}_* \hat{\mathscr{F}} \to v_{n*} R^q f_{n*} ((u_n^* \hat{\mathscr{F}}))$$

taking limit we get

$$R^q \hat{f}_* \hat{\mathscr{F}} \to \varprojlim_n R^q f_{n*} ((u_n^* \hat{\mathscr{F}})) \tag{22}$$

7.3 The comparison theorem and its corollaries

Theorem 7.3.1. *Let $f : X \to Y$ be a finte type morphism noetherian schemes, $Y' \subset Y$ a closed subscheme, $X' = f^{-1}(Y')$, $\hat{f} : \hat{X} \to \hat{Y}$ the extension of f to the formal completions of X, Y along X', Y'. Let $(\mathscr{F}_n = (u_n^* \hat{\mathscr{F}}) = i_n^* \mathscr{F}$ the sheaves on X_n. Then the morphisms* (21), (22)

$$\widehat{R^q f_* \mathscr{F}} \to R^q \hat{f}_*(\hat{\mathscr{F}}) \to \varprojlim_n R^q f_{n*}(\mathscr{F}_n)$$

are topological isomorphisms for all q.

Remark 7.3.2. *The above theorem is equivalent to the following special case.*

Theorem 7.3.3. *With the notation of the above theorem, suppose in addition that R is a Noetherian ring, $I \subset R$ an ideal with $Y = SpecR$, $Y' = SpecR/I$, $\mathcal{I} = \tilde{I}$, $Y_n = Spec\ R/I^{n+1}, X_n = Y_n \times_Y X$. Since $X' = f^{-1}(Y')$, X' is defined by the ideal sheaf $f^*(\mathcal{I})\mathcal{O}_X$. To simplify the notations, we identify this with $\mathcal{I}$ itself. Let $\mathscr{F}_n = i_n^* \mathscr{F} = \mathscr{F}/\mathcal{I}^{n+1}\mathscr{F}$. Then there are topological R-modules isomorphisms*

$$\widehat{H^q(X, \mathscr{F})} \to \varprojlim_n H^q(X, \mathscr{F}_n) \tag{23}$$

$$H^q(\hat{X}, \hat{\mathscr{F}}) \to \varprojlim_n H^q(X, \mathscr{F}_n) \tag{24}$$

where they come from base change maps and the commutative diagram

$$\begin{tikzcd} & \hat{X} \arrow[dr, "i"] & \\ X_n \arrow[ur, "u_n"] \arrow[rr, "i_n"] & & X \end{tikzcd}$$

(here $\widehat{H^q(X, \mathscr{F})}$ is endowed with the I-adic topology).

Corollary 7.3.4. *Let R be a noetherian ring, $I \subset R$ an ideal, $Y = SpecR$, $f : X \to Y$ a morphism of finite type, $X' = f^{-1}(Y')$, $Y' = SpecR/I$, $\hat{Y} = Spf\hat{R}$, $\hat{f} : \hat{X} \to \hat{Y}$ the extension of f to the formal completions of X, Y along X', Y'. Let $\mathscr{F}, \mathscr{G}$ be coherent sheaves on X whose supports intersect properly over Y. Then $\mathrm{Ext}^q(F, G)$ is a finitely generated R module and*

$$\widehat{Ext^q(F, G)} \simeq Ext^q(\hat{F}, \hat{G})$$

Proof. We need a version of base change map for complexes in derived category, namely the isomorphism

$$i^* Rf_* F \cong R\hat{f}_* i^* F$$

for $F \in D^+(X)$, where Rf_* means the total derived functor of f_*.

Consider the left exact functor $Hom(F, -) = \Gamma(X, \mathcal{H}om(F, -))$. Then

$$Ext^q(F, G) = H^q RHom(F, G) = H^q R\Gamma(X, R\mathcal{H}om(F, G)) = R^q f_* R\mathcal{H}om(F, G)(Y)$$

Similarly

$$Ext^q(\hat{F}, \hat{G}) = H^q RHom(\hat{F}, \hat{G}) = H^q R\Gamma(\hat{X}, R\mathcal{H}om(\hat{F}, \hat{G})) = R^q \hat{f}_* R\mathcal{H}om(\hat{F}, \hat{G})(\hat{Y})$$

By Compaison theorem

$$i^* R f_* R\mathcal{H}om(F, G) \cong R\hat{f}_* i^* R\mathcal{H}om(F, G)$$

from flatness of the map i we get

$$i^* R\mathcal{H}om(F, G) \cong R\mathcal{H}om(i^* F, i^* G) = R\mathcal{H}om(\hat{F}, \hat{G})$$

Therefore we get an isomorphsim in $D^+(\hat{X})$

$$i^* R f_* R\mathcal{H}om(F, G) \cong R\hat{f}_* R\mathcal{H}om(\hat{F}, \hat{G})$$

Take global section and then take the q-th cohomology of both sequence to obtain the result. □

Corollary 7.3.5. *In addition to the above assumptions, suppose R is adic. Then the completion functor*

$$\left\{ \begin{array}{c} \text{Coherent sheaves on } X \text{ with} \\ \text{proper support over } Y \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{Coherent sheaves on } \hat{X} \text{ with} \\ \text{proper support over } \hat{Y} \end{array} \right\}$$

$$F \longmapsto \hat{F}$$

is fully faithful.

Proof. We have $Ext^0(F, G) = Hom(F, G)$ which is a finitely generated R-module by the above corollary. Since R is complete and separated with respect to I-adic topology, so is $Hom(F, G)$. Again by the above corollary

$$Hom(F, G) = \widehat{Hom(F, G)} \cong Hom(\widehat{F}, \widehat{G}).$$

□

7.4 Proof of the comparison theorem

7.4.1 I-good filtration

Let R be a noetherian ring, I an ideal of R, M a finitely generated R module endowed with a decreasing filtration by submodules $(M_n)_{n \in \mathbb{Z}}$.

Definition 7.4.1. *The filtration (M_n) is called I-**good** if it satisfies the following:*

(i) there exists n_1 such that $M_{n_1} = M$.

(ii) $IM_n \subset M_{n+1}$ for all $n \in \mathbb{Z}$.

(iii) there exists an integer n_0 such that $M_{n+1} = IM_n$ for all $n \geq n_0$.

Example 7.4.2. *The I-adic filtration of M, defined by $M_n = M$ for $n \leq 0$ and $M_n = I^n M$ for $n \geq 0$ is I-good.*

Lemma 7.4.3. *All I-good filtrations on M define the same topology, namely the I-adic topology.*

Consider the graded ring

$$A' := \bigoplus_{n\in\mathbb{N}} I^n$$

and the graded module over A',

$$M' = \bigoplus_{n\in\mathbb{N}} M_n$$

Lemma 7.4.4. *Assume (i) in 7.4.1, then condition (iii) is equivalent to:*

M' is a finitely generated A'-module.

7.4.2 Artin-Rees lemma

ArtinRees lemma is a result about finitly generated modules over a Noetherian ring. It was proved independently by E. Artin and D. Rees in 1950s.

Theorem 7.4.5. *Let I be an ideal in a Noetherian ring R; let M be a finitely generated $R-$module and let N a submodule of M. Then there exists an integer $k \geq 1$ such that, for $n \geq k$,*

$$I^n M \cap N = I^{n-k}((I^k M) \cap N).$$

This simply means, the filtration induced on N by I-adic filtration of M is eventually stable, in other words, by setting $N_n = I^n M \cap N$, the decreasing sequence of submodules $N_0 \supset N_1 \supset N_2 \supset \cdots$ will be eventually stable as an I-filtration; $IN_n = N_{n+1}$ for n large.

7.4.3 Mittag-Leffler condition

Definition 7.4.6. *Let R be a commutative ring, and $M_\bullet = (M_n \xrightarrow{u_{mn}} M_m)$ be a projective system of R-modules, indexed by $\mathbb{N}$.*

*(i) $M_\bullet$ is **strict** if the transition maps u_{mn} are surjective.*

*(ii) $M_\bullet$ is **essentially zero** if for each m there exists $n \geq m$ such that $u_{mn} = 0$.*

*(iii) $M_\bullet$ satisfies the **Mittag-Leffler** condition (ML) if for each m there exists $n \geq m$ such that, for all $n' \geq n, \operatorname{Im} u_{mn'} = \operatorname{Im} u_{mn}$ in M_m.*

Lemma 7.4.7. *(1) If $M_\bullet$ is essentially zero, then $\lim_n M_n = 0$.*
(2) The functor$M_\bullet \mapsto \varprojlim_n M_n$ is left exact.
(3) Let

$$0 \to L_\bullet \to M_\bullet \to N_\bullet \to 0$$

be an exact sequence of inverse systems of R-modules; if $L_\bullet$ satisfies ML then the sequence

$$0 \to \varprojlim_n L_n \to \varprojlim_n M_n \to \varprojlim_n N_n \to 0$$

is exact.

(4) let

$$0 \to L_\bullet \to M_\bullet \to N_\bullet \to 0$$

be an exact sequence of inverse systems of R-modules. If $L_\bullet$ and $N_\bullet$ satisfies ML, so does $M_\bullet$.

Now we get to a technical lemma about the commutation of cohomology groups with inverse limits.

Theorem 7.4.8. *Let X be a scheme, and let $(F_n)_{n\in\mathbb{N}}$ be an inverse system of quasi-coherent sheaves on X with surjective transition maps. Assume that, for all $i \in \mathbb{Z}$, the inverse system (of $\mathbb{Z}$-modules) $H^i(X, F_\bullet)$ satisfies ML. Then, for all $i \in \mathbb{Z}$, the natural* map

$$H^i\left(X, \varprojlim_n F_n\right) \to \varprojlim_n H^i(X, F_n)$$

is an isomorphism.

7.4.4 The Proof

Grothendieck's original proof used descending induction on q. However, in the following proof, the integer q is fixed in the entire proof.

We have the short exact sequence

$$0 \to \mathcal{I}^{n+1}F \to F \to F_n \to 0$$

We deduce the long exact sequence of the cohomology groups

$$H^q(X, \mathcal{I}^{n+1}F) \to H^q(X, F) \xrightarrow{\phi} H^q(X, F_n) \to H^{q+1}(X, \mathcal{I}^{n+1}F) \to H^{q+1}(X, F)$$

Let $R_n = \operatorname{Ker}\phi$ and $Q_n = \operatorname{Coker}\phi$. So we have the short exact sequence

$$0 \to R_n \to H^q(X, F) \xrightarrow{\phi} H^q(X, F_n) \to Q_n \to 0 \tag{25}$$

The following lemma is the essence of the proof.

Lemma 7.4.9. *(1) The filtration (R_n) on $H^q(X, F)$ is I-good, therefore, the topology defined by (R_n) on $H^q(X, F)$ is the I-adic topology.*

(2) The inverse system $Q = (Q_n)$ is essentially zero.

(3) The inverse system $(H^q(X, F_n))_n$ satisfies the Mittag-Leffler condition.

Proof using the lemma

By (25) we have the exact sequence

$$0 \to H^q(X, F)/R_n \to H^q(X, F_n) \to Q_n \to 0$$

By 7.4.9(2) and 7.4.7 we have $\lim_n Q_n = 0$. So, By the left exactness of the functor $\varprojlim_n$ we get

$$0 \to \varprojlim_n H^q(X,F)/R_n \to \varprojlim_n H^q(X,F_n) \to 0$$

So, $\varprojlim_n H^q(X,F)/R_n \xrightarrow{\sim} \varprojlim_n H^q(X,F_n)$.

Since the filtration (R_n) is I-good(7.4.9(1)) we have

$$\widehat{H^q(X,F)} = \varprojlim_n H^q(X,F)/I^{n+1}H^q(X,F) \xrightarrow{\sim} \varprojlim_n H^q(X,F)/R_n \cong \varprojlim_n H^q(X,F_n)$$

Which proves the isomorphism (23).

We have

$$H^q(\hat{X},\hat{F}) = H^q\left(\hat{X}, \varprojlim_n F_n\right) = H^q\left(X, i_* \varprojlim_n F_n\right) = H^q\left(X, \varprojlim_n (i_n)_* F_n\right)$$

By 7.4.9(3) and Theorem 7.4.8 lim and cohomology commute, so

$$H^q\left(X, \varprojlim_n (i_n)_* F_n\right) = \varprojlim_n H^q(X,(i_n)_* F_n) = \varprojlim_n H^q(X,F_n)$$

Which completes the proof.

Proof of the lemma

It remains to show $(1),(2),(3)$ of 7.4.9

Proof of (1)**.** We should check the three conditions of an I-good filtration.

(i) we have $R_{-1} = H^q(X,F)$.

(ii) it follows from the map of graded $B = \bigoplus_{n\in\mathbb{N}} I^n$-module

$$\bigoplus_{n\in\mathbb{N}} H^q\left(X,\mathcal{I}^{n+1}F\right) \to \bigoplus_{n\in\mathbb{N}} H^q(X,F).$$

(iii) by lemma 7.4.4 it is equivalent to that $\oplus_n R_n$ be a finitely generated $B = \bigoplus_{n\in\mathbb{N}} I^n$-module. Apply corollary 7.1.11 to IF, $\bigoplus_{n\in\mathbb{N}} H^q(X,\mathcal{I}^{n+1}F)$ is a finitely generated graded B-module. Since $\oplus_n R_n$ is the homomorphic image of $\bigoplus_{n\in\mathbb{N}} H^q(X,\mathcal{I}^{n+1}F)$, it is also finitely generated.

Proof of (2)**.** This is the hardest part. For $n > m$ consider the commutative diagram

$$\begin{tikzcd}
H^q(X,F_n) \arrow[r] \arrow[d] & Q_n \arrow[r] \arrow[d, "u_{mn}"] & H^{q+1}(X,\mathcal{I}^{n+1}F) \arrow[d, "\phi"] \\
H^q(X,F_m) \arrow[r] & Q_m \arrow[r] & H^{q+1}(X,\mathcal{I}^{m+1}F)
\end{tikzcd}$$

By the finiteness theorem 7.1.11 if n is large enough we have$\phi(H^{q+1}(X,\mathcal{I}^{n+1}F)) \subset I^{n-m}H^{q+1}(X,\mathcal{I}^{m+1}F)$. So

$$u_{mn}(Q_n) \subset Q_m \cap \phi(H^{q+1}(X,\mathcal{I}^{n+1}F)) \subset Q_m \cap I^{n-m}H^{q+1}(X,\mathcal{I}^{m+1}F) \subset I^{n-m-r_0}(Q_m \cap I^{r_0}H^{q+1}(X,\mathcal{I}^{m+1}F)$$

for some $r_0 > 0$ and $n-m$ large enough(we used Artin-Rees Lemma 7.4.5 because $H^{q+1}(X,\mathcal{I}^{n+1}F)$ is a finitely generated R-module by Theorem 7.1.12). Because $\mathcal{I}^{m+1}F_m = 0$, $I^{m+1}H^q(X,F_m) = 0$ an R-module. Since

$$Q_m \;=\; Im(H^q(X,F_m) \to H^{q+1}(X,\mathcal{I}^{m+1}F))$$

therefore $I^{m+1}Q_m = 0$. Hence $u_{mn}(Q_n) = 0$ if $(n-m) \gg 0$.

Proof of (3)**.** It follows from (2). Take

$$0 \to H^q(X,F)/R_n \to H^q\,(X,F_n) \to Q_n \to 0$$

for all n. Transition maps of $H^q(X,F)/R_n$ are surjective, hence it trivially satisfies ML. Q_n is essentially zero by (2), so it satisfies ML too. By lemma 7.4.7 $(H^q\,(X,F_n)$) satisfies ML.

7.5 The Existence theorems

The existence theorems refer to some result about the question of algebraization; when is a formal scheme $\mathcal{X}$ a completion of a scheme X, $\hat{X} = \mathcal{X}$. Grothendieck approached this problem in analogous to what Serre did in GAGA, first fixing an scheme X and comparing the coherent sheaves, then attack the problem of algebraization.

7.5.1 Equivalence of categories and its corollaries

The fundamental theorem is the following, which we postpone the proof to the next sections.

Let R be an adic noetherian ring, I an ideal of definition of R, $Y = SpecR$, $Y_n = SpecR/I^{n+1}, \hat{Y} = SpfR$.

For a locally noetherian Y-scheme X, define its I-adic completion $\hat{X} = \varinjlim X_n$ where $X_n = X \times_Y Y_n$.

Theorem 7.5.1. *With the above notation, let X be a noetherian scheme, separated and of finite type over Y. Then the completion functor*

$$\left\{ \begin{array}{c} \textit{Coherent sheaves on X with} \\ \textit{proper support over Y} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \textit{Coherent sheaves on } \hat{X} \textit{ with} \\ \textit{proper support over } \hat{Y} \end{array} \right\}$$

$$F \quad \longmapsto \quad \hat{F}$$

is an equivalence of category.

We prove the theorem in the next sections. First we give some corollaries.

Corollary 7.5.2. *Let X be a Noetherian, separated, and of finite type morphism over Y. There is a bijection*

$$\left\{ \begin{array}{c} \textit{Closed subschemes of } X \\ \textit{which are proper over } Y \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \textit{Closed subschemes of } \hat{X} \\ \textit{which are proper over } \hat{Y} \end{array} \right\}$$

$$Z \longmapsto \hat{Z}$$

Proof. We prove the surjectivity which is the hard part. Let $\mathcal{Z} \subset \hat{X}$ be a closed formal subscheme of $\hat{X}$ which is proper over $\hat{Y}$. $\mathcal{Z}$ corresponds to a coherent sheaf $\mathcal{O}_{\mathcal{Z}}$ which has proper support over $\hat{Y}$. By the above theorem there exist a unique coherent $\mathcal{O}_X$-module sheaf F with proper support over Y such that $\hat{F} = \mathcal{O}_{\mathcal{Z}}$. Intersection of Supports of $\hat{F}$ and $\mathcal{O}_{\mathcal{Z}}$ is support of $\hat{F}$ which is proper over Y. Hence by corollary 7.3.4 for $q = 0$ the map $Hom(\mathcal{O}_X, F) \to Hom(\mathcal{O}_{\hat{X}}, \mathcal{O}_{\mathcal{Z}})$ is bijective. Therefore, there is a unique map $l : \mathcal{O}_X \to F$ with $\hat{l} = u$ for the surjection map $u : \mathcal{O}_{\hat{X}} \to \mathcal{O}_{\mathcal{Z}}$. $l_0 = u_0$ is surjection, so l is surjection. Therefore, there is an ideal sheaf $\mathcal{O}_Z$ corresponding to a closed subscheme $Z \subset X$ such that $F = \mathcal{O}_Z$. Hence $\hat{Z} = \mathcal{Z}$.

Corollary 7.5.3. *With above assumptions, there is an equivalence of categories*

$$\left\{ \begin{array}{c} \textit{Finite } X\textit{-schemes} \\ \textit{which are proper over } Y \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \textit{Finite } \hat{X}\textit{-schemes} \\ \textit{which are proper over } \hat{Y} \end{array} \right\}$$

$$Z \longmapsto \hat{Z}$$

Corollary 7.5.4. *Let X be a proper Y-scheme and let Z be a noetherian scheme, separated and of finite type over Y. Then*

$$Hom_Y(X, Z) \to Hom_{\hat{Y}}(\hat{X}, \hat{Z})$$
$$f \to \hat{f}$$

is bijective.

Corollary 7.5.5. *The functor*

$$\left\{ \textit{Proper } Y\textit{-schemes} \right\} \longrightarrow \left\{ \hat{Y}\textit{-formal schemes} \right\}$$

$$X \longmapsto \hat{X}$$

is fully faithful.

7.5.2 The algebraization problem: the projective case

Let R be an adic noetherian ring, I an ideal of definition of R, $Y = SpecR$, $Y_n = SpecR/I^{n+1}$, $\hat{Y} = SpfR$.

Definition 7.5.6. *Given a proper adic noetherian $\hat{Y}$-formal scheme $\mathcal{X}$, we say it is* ***algebraizable*** *if there exist a proper Y-scheme X such that $\hat{X} = \mathcal{X}$.*

Lemma 7.5.7. *If $\mathcal{X} = \hat{X}$ is algebraizable, then X is unique.*

Proof. It is a consequence of 7.5.5 which says the functor $X \mapsto \hat{X}$ is fully faithful.

It is not true that all formal schemes are algebraizable, however, for the projective case the following holds.

Theorem 7.5.8. *Let $\mathcal{X} = \varprojlim X_n$ be a proper, adic $\hat{Y}$-formal scheme with $X_n = \mathcal{X} \times_{\hat{Y}} Y_n$. Let $\mathscr{L}$ be an invertible $\mathcal{O}_{\mathcal{X}}$-module such that $L_0 = \mathscr{L} \otimes \mathcal{O}_{X_0}$ is ample (X_0 is projective over Y_0). Then $\mathcal{X}$ is algebraizable. And if X is a proper Y-scheme such that $\hat{X} = \mathcal{X}$, there exists an unique ample line bundle M on X with $\hat{M} = \mathscr{L}$.*

7.6 Proof of the existence theorem

Proof of Theorem 7.5.1

By corollary 7.3.5 the functor is fully faithful. It remains to prove it is essentially surjective.

We will prove it in three steps, first for the projective morphism, then for quasi-projective morphism, and lastly the general case.

Step 1-Projective case:

The proof is like the proof of GAGA third theorem 4.5.

Assume $f : X \to Y$ is projective. Let L be an ample line bundle on X. As before, for an $\mathcal{O}_X$-module M write its twisted form as $M(n) = M \otimes L^{\otimes n}$. Analogously define $N(n) = N \otimes_{\mathcal{O}_{\hat{X}}} \hat{L}^{\otimes n}$ for an $\mathcal{O}_{\hat{X}}$-module N. The following lemma is the essence of the proof.

Lemma 7.6.1. *If E is a coherent sheaf on $\hat{X}$, then there exist non-negative integer m, r and a surjection homomorphism*

$$\mathcal{O}_{\hat{X}}(-m)^r \to E$$

Proof. See [17]. □

Now let E be a coherent sheaf on$\hat{X}$ with proper support on $\hat{Y}$. By the above lemma there exist an exact sequence

$$K \to \mathcal{O}_{\hat{X}}(-m_0)^{r_0} \to E \to 0$$

Where K is the kernel of the surjection map, which is coherent. Again, apply the lemma to K to get an exact sequence

$$\mathcal{O}_{\hat{X}}(-m_1)^{r_1} \xrightarrow{\theta} \mathcal{O}_{\hat{X}}(-m_0)^{r_0} \to E \to 0 \tag{26}$$

Since the completion functor is fully faithful, there exist a unique map $\eta : \mathcal{O}_X(-m_1)^{r_1} \xrightarrow{\theta} \mathcal{O}_X(-m_0)^{r_0}$ such that $\hat{\eta} = \theta$. Take $F = Coker\ \eta$. So we have

$$\mathcal{O}_X(-m_1)^{r_1} \xrightarrow{\eta} \mathcal{O}_X(-m_0)^{r_0} \to F \to 0 \tag{27}$$

The completion functor is exact (6.6.6), apply it to (27)

$$\mathcal{O}_{\hat{X}}(-m_1)^{r_1} \xrightarrow{\theta} \mathcal{O}_{\hat{X}}(-m_0)^{r_0} \to \hat{F} \to 0 \tag{28}$$

By comparing (26) and (28) we get $\hat{F} = E$.

2-Quasi-projective case:

Assume $j : X \to Y$ is quasi-projective, i.e, there is an open immersion $j : X \to Z$, where Z is projective over Y. Let E be a coherent sheaf on $\hat{X}$ with proper support on $\hat{Y}$. Then the extension by zero $\hat{j}_! E$ is coherent on $\hat{Z}$. So by step 1 there exist a coherent sheaf F on Z with $\hat{F} = \hat{j}_! E$. We have $Supp\ F \subset X$, hence $\widehat{(j^* F)} = E$.

3-General case:

We use noetherian induction on X, it means we should prove the following:

Noetherian induction assumption: For every closed subscheme T of X which its underlying spce is strictly contained in X, for all cohernt sheaves G on $\hat{T}$ with proper support on $\hat{Y}$, there is a coherent sheaf F with proper support over Y, such that $\hat{F} = G$.

Noetherian induction claim: Then, every coherent sheaf E on $\hat{X}$ with proper support over $\hat{Y}$ is algebraizable, i.e., there is a coherent sheaf F on X with proper support over Y, such that $\hat{F} = E$.

Apply the chow's lemma 7.1.1 to $X \to Y$ to get

$$Z \xrightarrow{g} X \xrightarrow{f} Y \tag{29}$$

where g is projective (hence proper) and surjective and fg is quasi-projective. There exist an open $U \subset X$ where induced map $g : g^{-1}(U) \to U$ is an isomorphism.

Let $T = X - U$ the closed subscheme and $\mathcal{J} \subset \mathcal{O}_X$ the ideal defining T. Using adjunction and the identity map $\hat{g}^* E \to \hat{g}^* E$ we get a map $E \to \hat{g}_* \hat{g}^* E$. Take its kernel and cokernel to get a short exact sequence

$$0 \to K \to E \to \hat{g}_* \hat{g}^* E \to C \to 0$$

The following lemma completes the proof.

Lemma 7.6.2. *(i) $\hat{g}_* \hat{g}^* E$ is algebraizable.*

(ii) K, C are algebraizable coherent sheaves.

(iii) Kernel, Cokerel, and extension of algebraizable sheaves are algebraizable, hence E is algebraizable, since it is an extension of K and $ker(\hat{g}_*\hat{g}^*E \to C)$.

Sketch of the proof. (i) g is proper, so $\hat{g}^*E$ has proper support over $\hat{Y}$. By chow's lemma $Z \xrightarrow{fg} Y$ is quasi-projective. By part 2, $\hat{g}^*E$ is algebraizable, lets say $\hat{L} = \hat{g}^*E$. By Comparison theorem we have (for q=0) $\hat{g}_*\hat{g}^*E = \hat{g}_*(\hat{L}) = \widehat{g_*L}$. So $\hat{g}_*\hat{g}^*E$ is algebraizable.

(ii)First we prove K, C are killed by some $\hat{\mathcal{J}}^n$ by working locally and replace X with $Spec\ B$ with IB an ideal of definition of B. Therefore, K, C can be seen as coheret sheaves on $\hat{T}'$ which is thickening of T by $\mathcal{J}^n$. Hence, by noetherian induction assumption, K, C are algebraizable.

(iii) Kernel and cokernel being algebraizable are obvious. The case of Extension follows from the corollary 7.2.4 of the comparison theorem for $r = 1$ (by the correspoendece of extensions and $Ext^1(A, B)$).

$\square$

8 The unified GAGA theorem

So far we have studied two GAGA theorems. The first one, which is the original GAGA theorem, associates a complex analytic space to a proper scheme X over $\mathbb{C}$. It was achieved by endowing the closed points of X with the complex topology and equipping it with the sheaf of holomorphic functions. We may call this analytification an "archimedean analytification". We give it this name since there are non-archimedean versions of GAGA theorems, like in rigid geometry, which work with the schemes over a non-archimedean field.

The second GAGA theorem was the formal GAGA. In this case, the analytification space was achieved by forming the formal completion of an scheme X along one of its closed subscheme $Z \subset X$, with the sheaf of functions of formal power series along Z.

The point is that the proof of all theorems are very much a like. First, we prove the GAGA theorems for the structure sheaf $\mathcal{O}_X$, then we prove it for all line bundles and the projective case, and finally use theorems like Chow's lemma to reduce the general case to the projective case. However in the first step, to prove the results for the structure sheaf, we usually make use of some deep theorems like the Oka coherence theorem and the finiteness of sheaf cohomology groups .

Jack Hall in [24] proved a vast generalization of the GAGA theorems and showed that all of these versions of GAGA can be put into a single framework, and they are special cases of a more general theorem. The idea is that those structure sheaves $\mathcal{O}_X$ somehow "generate" the category of coherent sheaves and the GAGA theorems are the consequence of a fact that some functor is *representable*. However, this proof is formulated in the language of derived and triangulated categories.

Recently, Amnon Neeman, inspired by the ideas of J. Hall, in [25] proved a more general and more powerful theorem with a shorter proof using the approximable triangulated categories and a more general representability theorem.

Our goal in this section is to present the proof of a special case of J. Hall theorems. One of the result of J. Hall in [24] is as follows.

Theorem 8.0.1. *Let R be a Noetherian ring. Let X be a proper scheme over Spec R. Let $c : \mathcal{X} \to X$ be a morphism of locally G-ringed spaces. Let X_{cl} be the set of closed points of X and let $\mathcal{X}_{\mathrm{cl},c} = c^{-1}\left(X_{\mathrm{cl}}\right)$. Assume that*
(1) $\mathcal{O}_{\mathcal{X}}$ is coherent;
(2) if $\mathcal{F} \in \mathrm{Coh}(\mathcal{X})$, then $\oplus_i \mathrm{H}^i(\mathcal{X}, \mathcal{F})$ is a finitely generated R-module;
(3) $c : \mathcal{X}_{\mathrm{cl},c} \to X_{\mathrm{cl}}$ is bijective;
(4) if $\mathcal{F} \in \mathrm{Coh}(\mathcal{X})$ and $\mathcal{F}_x = 0$ for all $x \in \mathcal{X}_{\mathrm{cl},c}$, then $\mathcal{F} = 0$;
(5) if $x \in \mathcal{X}_{\mathrm{cl},c}$, then $\mathcal{O}_{X,c(x)} \to \mathcal{O}_{\mathcal{X},x}$ is flat and $\kappa(c(x)) \to \kappa(c(x)) \otimes_{\mathcal{O}_{X,c(x)}} \mathcal{O}_{\mathcal{X},x}$ is an isomorphism.

Then the comparison map:
$$\mathrm{H}^i(X, F) \to \mathrm{H}^i\left(\mathcal{X}, c^* F\right)$$

is an isomorphism for all coherent sheaves F on X and

$$c^* : \mathrm{Coh}(X) \to \mathrm{Coh}(\mathcal{X})$$

is an exact equivalence of abelian categories.

Remark 8.0.2. *We don't give the definition of a G-ringed space, since we won't prove the general theorem. A G-ringed space is a set with a G-topology equipped with a sheaf of rings. G-topology is the generalization of the usual notion of topology on a set and it is a kind of Grothendieck topology.*

Remark 8.0.3. *Note that in the case of usual GAGA theorem, the condition (1) is the Oka coherence theorem 3.1.2, the condition (2) is the Cartan and Serre finiteness theorem 3.3.1, the condition (5) is the theorem 4.1.15 and the other conditions are obvious from the definition of the analytification space.*

We prove the theorem 8.0.1 in the special case of R being a field and $X \to Speck$ being a smooth projective morphism. Moreover, we assume $\mathcal{X}$ is a locally ringed space and the map c is flat and bijective on closed points. It is clear that the usual GAGA theorems lie in this special case. The proof of general theorem is the same but needs more powerful representability theorems. The proof of this case will illustrate the general strategy.

8.1 Preliminaries

8.1.1 Derived category and Triangulated category

In this section we briefly recall the definition of a derived category and the triangulated category.

The category of complexes

Let A be an abelian category. We can form the category of complexes $C(A)$. The objects of $C(A)$ are chain complexes and the morphisms are chain maps. It can be shown that $C(A)$ is an abelian category.

We can do two things with complexes; first is to shift a complex, second is to compute the cohomology objects. If $X^\bullet \in C(A)$, then $X^\bullet[1]$ is the complex with $(X^\circ[1])^i := X^{i+1}$ and differential $d^i_{X[1]} := -d^{i+1}_X$.

The shift $f[1]$ of a morphism of complexes $f : X^\bullet \to Y^\bullet$ is the complex morphism $X^\bullet[1] \to Y^\bullet[1]$ given by $f[1]^i := f^{i+1}$.

It defines a functor T, which we call the shift functor

$$T : \mathrm{C}(\mathcal{X}) \to \mathrm{C}(\mathcal{X}), X^\bullet \mapsto X^\bullet[1]$$

It can be easily seen that T defines an equivalence of abelian categories with the inverse functor T^{-1} given by $X^\bullet \mapsto X^\bullet[-1]$, where $X^\bullet[k]^i = X^{k+i}$ and $d^i_{X[k]} = (-1)^k d^{i+k}_X$ for any $k \in \mathbb{Z}$.

The cohomology $H^i(X^\bullet)$ of a complex $X^\bullet$ with the differential d is the quotient

$$H^i\left(X^\bullet\right) := \frac{\operatorname{Ker}\left(d^i\right)}{\operatorname{Im}\left(d^{i-1}\right)} \in \mathcal{A}$$

Any complex morphism $f : X^\bullet \to Y^\bullet$ induces natural homomorphisms

$$H^i(f) : H^i\left(X^\bullet\right) \longrightarrow H^i\left(Y^\bullet\right).$$

The induced map for the cohomology objects is used to define the notion of quasi-isomorphism, which plays an essential role in the derived category.

Definition 8.1.1. *A morphism of complexes $f : X^\bullet \to Y^\bullet$ is a quasi-isomorphism if for all $i \in \mathbb{Z}$ the induced map $H^i(f) : H^i\left(X^\bullet\right) \to H^i\left(Y^\bullet\right)$ is an isomorphism.*

Definition 8.1.2. *Let $f : X^\bullet \to Y^\bullet$ be a complex morphism. Its **mapping cone** is the complex $\mathrm{C}(f)$ with*

$$\mathrm{C}(f)^i = X^{i+1} \oplus Y^i \quad \textit{and} \quad d^i_{\mathrm{C}(f)} := \begin{pmatrix} -d^{i+1}_X & 0 \\ f^{i+1} & d^i_Y \end{pmatrix}$$

It can be shown the mapping cone is a complex. Moreover, There exist two natural complex morphisms

$$\tau : Y^\bullet \longrightarrow \mathrm{C}(f) \quad \text{and} \quad \pi : \mathrm{C}(f) \longrightarrow X^\bullet[1]$$

given by the natural injection $Y^i \to X^{i+1} \oplus Y^i$ and the natural projection $X^{i+1} \oplus Y^i \to X^\bullet[1]^i = X^{i+1}$. Also the

$$Y^\bullet \to \mathrm{C}(f) \to X^\bullet[1]$$

is a short exact sequence of complexes. In particular, we obtain the long exact cohomology sequence

$$H^i\left(X^\bullet\right) \longrightarrow H^i\left(Y^\bullet\right) \longrightarrow H^i(\mathrm{C}(f)) \longrightarrow H^{i+1}\left(X^\bullet\right) \longrightarrow \cdots$$

It can be shown that:

Lemma 8.1.3. *Any commutative diagram*

$$\begin{tikzcd} X_1^\bullet \arrow[r, "f_1"] \arrow[d] & Y_1^\bullet \arrow[r] \arrow[d] & \mathrm{C}(f_1) \arrow[r] \arrow[d, dotted] & X_1^\bullet[1] \arrow[d] \\ X_2^\bullet \arrow[r, "f_2"] & Y_2^\bullet \arrow[r] & \mathrm{C}(f_2) \arrow[r] & X_2^\bullet[1]. \end{tikzcd}$$

can be completed, i.e, the dotted morphism exist such that the diagram becomes commutative.

The main use of the mapping cone is to identify quasi-isomorphisms:

Theorem 8.1.4. *Let A be an abelian category and $X^\bullet, Y^\bullet \in C(A)$ be two chain complexes. A chain morphism $f : X \to Y$ is quasi-isomorphism if and only if $Cone(f)$ is acyclic, i.e, all its cohomology objects vanish.*

Proof. We write the long exact sequence of cohomology of the sequence $X^\bullet \xrightarrow{f} Y^\bullet \to \mathrm{C}(f) \to X^\bullet[1]$ and the result follows from the fact that all the cohomology of $\mathrm{C}(f)$ vanish. $\square$

The Derived category

The central idea for the definition of the derived category is that to make quasi-isomorphic complexes isomorphic. It requires that the quasi-isomorphisms to be invertible. To construct such a category we should somehow "localize" the quasi-isomorphisms. We don't go through the construction of the derived category, the reader can see the details in [21].

Theorem 8.1.5. *Let A be an abelian category and $C(A)$ be its category of complexes. There exist a category $D(A)$, the derived category of A, and the functor*

$$Q : C(A) \to D(A)$$

such that :
1) if the morphism f is a quasi-isomorphism, then $Q(f)$ os an isomorphism.
2)Any functor $F : C(A) \to B$ satisfying the property 1 factorizes uniquely over $Q : C(A) \to D(A)$, i.e, there exist a unique functor $G : D(A) \to B$ with $F \simeq G \circ Q$.

We have the following facts:

Corollary 8.1.6. *1) Under the functor $Q : \mathrm{C}(\mathcal{A}) \to \mathrm{D}(\mathcal{A})$ the objects of the two categories $\mathrm{C}(\mathcal{A})$ and $\mathrm{D}(\mathcal{A})$ are identified.*

2) The cohomology objects $H^i(X^\bullet)$ of an object $X^\bullet \in \mathrm{D}(\mathcal{A})$ are well-defined objects of the abelian category $\mathcal{A}$.

3) Viewing any object in $\mathcal{A}$ as a complex concentrated in degree zero yields an equivalence between $\mathcal{A}$ and the full subcategory of $\mathrm{D}(\mathcal{A})$ that consists of all complexes $X^\bullet$ with $H^i(X^\bullet) = 0$ for $i \neq 0$.

Since the quasi-isomorphism become isomorphisms in the derived category, the zero map gives an isomorphism of an acylic complex to the zero complex.

Remark 8.1.7. *A complex $X^\bullet$ is isomorphic to 0 in $\mathrm{D}(\mathcal{A})$ if and only if all the cohomology objects vanish.*

Remark 8.1.8. *Contrary to the category of complexes $C(\mathcal{A})$, the derived category $\mathrm{D}(\mathcal{A})$ is in general not abelian, but it is an additive category.*

We said that the objects of $\mathrm{D}(\mathcal{A})$ can be identified with the the complexes in $\mathrm{C}(\mathcal{A})$. The morphisms in this category is harder to define, they are diagrams of the form

$$\begin{tikzcd} & Z^\bullet \arrow[dl, "qis"'] \arrow[dr] & \\ X^\bullet & & Y^\bullet \end{tikzcd}$$

where $Z^\bullet \to X^\bullet$ is a quasi-isomorphism. One has to define when two such morphisms are equivalent and how to compose them, but we don't want to repeat all the details and we wont use them either.

The complexes in the categories $C(A)$ and $D(A)$ may be unbounded, but it is convenient to work with the bounded ones. Let $C(A)^*$ with $* = +, -, b$ be the category of complexes X^* with $X^i = 0$ for $i \ll 0$, $i \gg 0$, and $|i| \gg 0$ respectively. Let $D(A)^*$ for $* = +, -, b$ be the corresponding subcategories of $D(A)$. It can be shown that the natural functors $D(A)^* \to D(A)$ define equivalences of $D(A)^*$ with the full subcategories of all complexes $X^\bullet \in D(A)$ with $H^i(X^\bullet) = 0$ for $i \ll 0$, $i \gg 0$, and $|i| \gg 0$ respectively.

Triangulated structure

So far, we have defined the shift functor $[1] : D(A) \to D(A)$, and the mapping cone $C(f)$ for a morphism $f^\bullet : X^\bullet \to Y^\bullet$. With these two notions, $D(A)$ has the structure of a **triangulated category**.

A triangulated category is a category with the additional structure of a "shift functor" and a class of "exact triangles". The exact triangles generalize the short exact sequences in an abelian category. The most important examples of a triangulated category is the derived category.

A **shift** on a category D is an additive automorphism $[1] : D \to D$. A triangle (X, Y, Z, u, v, w) consists of datas in the sequence $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$.

A **triangulated category** is an additive category D with a shift functor and a class of triangles, called exact triangles satisfying some properties called $TR1, TR2, TR3, TR4$.

$TR1$ has several property: The first property is that for every opbject X, the triangle $X \xrightarrow{id}\to X \to 0 \to X[1]$ is exact. The second property is that for every morphism $u : X \to Y$ in our category, there is an object Z called a **cone** of u fitting into an exact triangle $X \xrightarrow{u}\to Y \to Z \to X[1]$. It is clear that the mapping cone in the derived category satisfies this condition. The third property is that every triangle isomorphic to an exact triangle is exact.

The condition $TR2$ says that if $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$ is an exact triangle, then the shifted triangles $Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u[1]} Y[1]$ and $Z[-1] \xrightarrow{-W[-1]} X \xrightarrow{u} Y \xrightarrow{v} Z$ are exact. $TR3$ is the similar condition to lemma 8.1.3. It means that given two exact triangles, for a morphism between the first morphisms in each triangle, there exist a morphism between the third objects of two triangles which makes everything commute. Finally the $TR4$ property is rather involved, it is called the" octahedron axiom", It roughly says that this: Let $u : X \to Y$ and $v : Y \to Z$ be morphisms, consider the composed morphism $vu : X \to Z$. By condition $TR1$, form exact triangles for each of these three morphisms. The octahedral axiom states that the three mapping cones can be made into the vertices of an exact triangle so that "everything commutes".

Example 8.1.9. *clearly, the derived category $D(A)$ of an abelian category A is a triangulated category with the usual notion of shift functor and the mapping cones.*

Triangulated categories admit a notion of cohomology, and every triangulated category has a large supply of cohomological functors.

Definition 8.1.10. *A **cohomological functor** F from a triangulated category D to an abelian category A is a functor such that for every exact triangle*

$$X \to Y \to Z \to X[1]$$

the sequence $F(X) \to F(Y) \to F(Z)$ in A is exact.

Since an exact triangle determines an infinite sequence of exact triangles,

$$\cdots \to Z[-1] \to X \to Y \to Z \to X[1] \to \cdots$$

a cohomological functor F gives a long exact sequence in the abelian category A:

$$\cdots \to F(Z[-1]) \to F(X) \to F(Y) \to F(Z) \to F(X[1]) \to \cdots$$

.

For example each object B in a triangulated category D, the functors $\mathrm{Hom}(B,\text{-})$ and $\mathrm{Hom}(\text{-},B)$ are cohomological functors, with values in the category of abelian groups.

An **exact functor** from a triangulated category D to a triangulated category E is an additive functor which commutes with shift and sends exact triangles to exact triangles.

Much of the results in a derived category are the consequence of the shift functor on complexes and the mapping cones, therefore most of the results holds in generality in a trianulated category. It is easier to work with trianulated category, since the construction of the derived category of an abelian category is involved. Different abelian categories can have equivalent derived categories, so that it is not always possible to reconstruct A from $D(A)$ as a triangulated category. This situation can be described by the notion of a *t-structure* on a triangulated category. Since we wont use them, we don't give the definition.

Now we give some examples in the category of sheaves.

Example 8.1.11. *Let $(X, \mathcal{O}_X)$ be locally ringed space. Let $D^b(Coh(\mathcal{O}_X))$ be the bounded derived category of the coherent $\mathcal{O}_X$-modules. Let $D^b_{Coh}(\mathcal{O}_X)$ be the full triangulated subcategory of $D(\mathcal{O}_X$-modules) of bounded complexes with coherent cohomology sheaves.*

Lemma 8.1.12. *If X is a Noetherian scheme, then the canoncial functor $D^b(Coh(\mathcal{O}_X)) \to D^b_{Coh}(\mathcal{O}_X)$ is an equivalence of category.*

Relation to Derived functors

The derived category is a natural framework to define and study derived functors, as the name "derived" is shared by both of them. In the following, we will deal with right derived functors coming from left exact functors. The left derived functors can be described similarly.

Let $F : A \to B$ be a left exact functor between two abelian categories with enough injectives. Typical examples are $F : AAb$ given by $X \mapsto Hom(X, O)$ or $X \mapsto Hom(O, X)$ for some fixed object O, or the global sections functor on sheaves or the direct image functor.

We know that there are derived functors $R^nF : A \to B$ for all $n \geq 1$. The derived category allows us to encapsulate all derived functors R^nF in one functor, namely the **total derived functor** $RF : D^+(A) \to D^+(B)$. The classical derived functors are related to the total one by $R^nF(X) = H^n(RF(X))$ by viewing an object $X \in A$ as a complex in $D^+(A)$ concentrated in degree zero. Roughly speaking, the usual derived functors R^nF only keep the cohomologies while the total derived functor RF keeps track of the complexes as well.

Example 8.1.13. ***(Cohomology)***
Let $(X, \mathcal{O}_X)$ be a ringed space. We know that the category of $\mathcal{O}_X$-modules have enough injectives. The globale sectio functor $\Gamma : Mod(X) \to Ab$ is left exact. Hence its right derived functor

$$R\Gamma : D^+Mod(X) \to D^+Ab$$

exist. The higher derived functors are denoted $H^i(X, \mathcal{F}^\bullet) = R^i\Gamma(\mathcal{F}^\bullet)$. For a sheaf they are just cohomology groups, in general they are called hypercohomology groups.

In the case of a Noetherian scheme X over a field k and the category of quasi-coherent sheaves, the functor $R\Gamma$ induces an exact functor

$$R\Gamma : D^b(Qcoh(X)) \to D^b(Vec(k))$$

This follows from a theorem of Grothendieck that the cohomology groups $H^i(X, \mathcal{F}) = 0$ for $i > dim(X)$ and a quasi-coherent sheaf $\mathcal{F}$.

Example 8.1.14. ***(Tensor product)***
Note that the tensor functor (for example in the category of R-modules) is not a left exact functor but a right exact functor. Therefor for getting derived functor we need a projective resolution rather than an injective resolution. The category of Rmodules have enough projective as every free R-module is projective, therefore for the tensor functor $T(A) = A \otimes_R B$, the left exact functors $Tor_i^R(A, B)$ exists. Now let us consider the category of coherent sheaves $Coh(X)$. Any coheret sheaf $\mathcal{F}$ admits a resolution $\mathcal{E}^\bullet \to \mathcal{F}$ by locally free sheaves and if $\mathcal{E}^\bullet$ is acyclic complex bounded above with locally free shaves $\mathcal{F} \otimes \mathcal{E}^\bullet$ is still acyclic. Therefore the right exact functor $\mathcal{F} \otimes (-)$ has a left derived functor

$$\mathcal{F} \otimes^L () : D^-(X) \to D^-(X)$$

In general the category of $\mathcal{O}_X$-modules has enough flat objects and there is a derived tensor product

$$- \otimes^L - : D^b(X) \times D^b(X) \to D^b(X)$$

Example 8.1.15. ***(Inverse image)***
For a morphism of ringed spaces $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$, the inverse image functor f^* is compostion of exact functor f^{-1} and the right exact functor $\mathcal{O}_X \otimes_{f^{-1}(\mathcal{O}_Y)} (-)$, Hence f^* is right exact. If $\mathcal{O}_X \otimes^L_{f^{-1}(\mathcal{O}_Y)} (-)$ is the left derived functor of $\mathcal{O}_X \otimes_{f^{-1}(\mathcal{O}_Y)} (-)$, then

$$Lf^* = (\mathcal{O}_X \otimes^L_{f^{-1}(\mathcal{O}_Y)} (-)) \circ f^{-1}$$

is the left total derived functor of f^*.

Note that if the morphism f is flat, then f^* becomes exact and we don't need the derived functors.

Example 8.1.16. ***(Local Homes)***
Recall that the Hom sheaf $\mathcal{H}om(\mathcal{F}, \mathcal{E})$ of two $\mathcal{O}_X$-modules is the sheaf $U \mapsto Hom(\mathcal{F}|_U, \mathcal{E}|_U)$. The functor $\mathcal{H}om(\mathcal{F}, -)$ is left exact, so the total derived functor

$$R\mathcal{H}om(\mathcal{F}, -) : D^+(Mod(X)) \to D^+(Mod(X))$$

exists.

8.1.2 Perfect complexes and Duals

Definition 8.1.17. *Let $(X, \mathcal{O}_X)$ be a locally ringed space. An $\mathcal{O}_X$-module $\mathcal{F}$ is called a locally free module if every point of $x \in X$ has a neighborhood U such that $\mathcal{F}|_U$ is isomorphic to $\bigoplus_i \mathcal{O}_X|_U$ as an mathcal$O_X|_U$-module.*

Definition 8.1.18. *Let $(X, \mathcal{O}_X)$ be a locally ringed space. Let $C^\bullet$ be a complex of $\mathcal{O}_X$-modules. We say $C^\bullet$ is **strictly perfect** if C^i is zero for all but finitely many i and C^i is a direct summand of a finite locally free $\mathcal{O}_X$-module for all i.*

Lemma 8.1.19. *Some properties of strictly perfect complexes:*

1) The cone on a morphism of strictly perfect complex is strictly perfect.

2)The total complex associated to the tensor product of two strictly perfect complexes is strictly perfect.

3)Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. If $\mathcal{F}^\bullet$ is a strictly perfect complex of $\mathcal{O}_Y$-modules, then $f^*\mathcal{F}^\bullet$ is a strictly perfect complex of $\mathcal{O}_X$-modules.

Definition 8.1.20. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $C^\bullet$ be a complex of $\mathcal{O}_X$-modules. We say $C^\bullet$ is **perfect** if there exists an open covering $X = \bigcup U_i$ such that for each i there exists a morphism of complexes $C_i^\bullet \to C^\bullet|_{U_i}$ which is a quasi-isomorphism with $C_i^\bullet$ a strictly perfect complex of $\mathcal{O}_{U_i}$-modules. An object E of $D(\mathcal{O}_X)$ is called perfect if it can be represented by a perfect complex of $\mathcal{O}_X$-modules.*

Lemma 8.1.21. *Let $X = Spec\ R$ be an affine scheme. Let $M^\bullet$ be a complex of R-modules and let E be the corresponding object of $D(\mathcal{O}_X)$. Then E is a perfect object of $D(\mathcal{O}_X)$ if and only if $M^\bullet$ is perfect as an object of $D(R)$.*

Lemma 8.1.22. *Let R be a regular ring of finite dimesion. Then a complex of R-modules $K^\bullet$ is perfect if and only if $K^\bullet \in D^b(R)$ and each $H^i(K^\bullet)$ is a finite R-module.*

From the two previous lemmas, and since the notion of perfectness is local, it follows that if the scheme X has some nice properties, then every object of $D^b(Coh(\mathcal{O}_X))$ is perfect.

Theorem 8.1.23. *Let X be a Noetherian regular scheme of finite dimension. Then every object of $D^b_{Coh}(\mathcal{O}_X))$ is perfect and conversely every perfect object of $D(\mathcal{O})$ is in $D^b_{Coh}(\mathcal{O}_X)))$.*

Lemma 8.1.24. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let K be a perfect object of $D(\mathcal{O}_X)$. Then $K^\vee = \mathrm{R}_{Hom}(K, \mathcal{O}_X)$ is a perfect object and $(K^\vee)^\vee \simeq K$. There are functorial isomorphisms*

$$M \otimes^L_{\mathcal{O}_X} K^\vee = R\mathcal{H}om(K, M)$$

and

$$\mathcal{H}^0(\mathrm{R}\Gamma\left(\mathcal{X}, M \otimes^L_{\mathcal{O}_X} K^\vee\right)) \simeq Hom_{D(\mathcal{O}_X)}(K, M)$$

Proof. See [7] lemma 20.47.5. □

Note that in the above we lemma we are working over the derived category $D(\mathcal{O}_X)$. Sometimes we are weoking with bounded complexs?????

8.1.3 The Bondal and Van Den Bergh representability theorem

The following representability theorem was first appeared in [22]. But later A. Bondal and M. Van Den Bergh in [23] gave a short proof of its generalization. Since we don't need that generalization, we aren't going to state it.

First we need a definition. Let k be a field and all categories are assumed to be k-linear.

Definition 8.1.25. *Let $\mathcal{D}$ be a triangulated category with the shift functor* [1]. *We say a cohomological functor $H : \mathcal{D} \to Vect(k)$ is* ***of finite type*** *if for all object $A \in \mathcal{D}$,*

$$\Sigma_i dim(H(A[i])) < \infty$$

Theorem 8.1.26. *Asumme that X is a regular projective variety over the field k. Let $D^b_{Coh}(X)$ be the derived category of bounded complexes of $\mathcal{O}_X$-modules with coherent cohomology sheaves. Then every contravariant cohomological functor of finite type is representable.*

Proof. See [23]. □

This theorem is our main ingredient to prove that the derived pullback c^* of the map c in theorem 8.0.1 admits a right adjoint c_*.

8.1.4 Projection formula

In this section we want to state variants of the projection formula, and in the end state the one which we will use in the proof of our desired unified GAGA theorem involving perfect complexes. We follow [7]. the Porjection formula section.

Lemma 8.1.27. *Let $(X, \mathcal{O}_X)$ be a ringed space. Let $\mathcal{I}$ be an injective $\mathcal{O}_X$-module. Let $\mathcal{E}$ be an $\mathcal{O}_X$-module. Assume $\mathcal{E}$ is finite locally free on X, Then $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{I}$ is an injective $\mathcal{O}_X$-module.*

Lemma 8.1.28. *Let $f : X \to Y$ be a morphism of ringed spaces. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let $\mathcal{E}$ be an $\mathcal{O}_Y$-module. Assume $\mathcal{E}$ is finite locally free on Y. Then there exist an isomorphism*

$$\mathcal{E} \otimes_{\mathcal{O}_Y} Rf_*\mathcal{F} \longrightarrow Rf_*\left(f^*\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}\right)$$

in $D^+(Y)$ functorial in $\mathcal{F}$

Proof. We choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$. The pullback of a finite locally free sheaf is also of the same kind, so $f^*\mathcal{E}$ is finite locally free. By lemma 8.1.27 we get the injective resolution

$$f^*\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow f^*\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{I}^\bullet$$

Apply f_* to get

$$Rf_*\left(f^*\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}\right) = f_*\left(f^*\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{I}^\bullet\right)$$

Hence it suffices to show $f_*\left(f^*\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}\right) = \mathcal{E} \otimes_{\mathcal{O}_Y} f_*(\mathcal{F})$. First we prove this for $\mathcal{E} = \mathcal{O}_Y^{\oplus n}$, then for the finite locally free sheaf $\mathcal{E}$.

□

Now we want to generalize this formula to complexes in the derived category of $D(\mathcal{O}_X)$.

Let $f : X \to Y$ be a morphism of ringed spaces. Let $E \in D(\mathcal{O}_X)$ and $K \in D(\mathcal{O}_Y)$. Then there is a map

$$Rf_*E \otimes^{\mathbf{L}}_{\mathcal{O}_Y} K \longrightarrow Rf_*\left(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*K\right)$$

which is the adjoint to the canonical map

$$Lf^*\left(Rf_*E \otimes^{\mathbf{L}}_{\mathcal{O}_Y} K\right) = Lf^*Rf_*E \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*K \longrightarrow E \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*K$$

The general version of the projection formula is the following.

Lemma 8.1.29. *Let $f : X \to Y$ be a morphism of ringed spaces. Let $E \in D(\mathcal{O}_X)$ and $K \in D(\mathcal{O}_Y)$. If K is perfect, then*

$$Rf_*E \otimes^{\mathbf{L}}_{\mathcal{O}_Y} K = Rf_*\left(E \otimes^{\mathbf{L}}_{\mathcal{O}_X} Lf^*K\right)$$

in $D(\mathcal{O}_Y)$.

Sketch of the proof. It suffices to prove the statement locally, since Hence, by definition of perfect objects, we may assume K is represented by a strictly perfect complex of $\mathcal{O}_Y$-modules. The statement is true for $K = K_1 \oplus K_2$, if and only if the statement is true for K_1 and K_2. So we assume K is a finite complex of finite free $\mathcal{O}_Y$-module. The statement is invariant under direct summand in K, so it further reduces to the case $K = \mathcal{O}_Y[n]$, which is trivial.

□

We will use a similar projection formula in the derived category of sheaves with coherent cohomology in the proof of our unified GAGA theorem.

8.1.5 Derived Nakayama's lemma

We need a version of the Nakayama lemma as follows.

Lemma 8.1.30. *Let (R, m) be a Noetherian local ring and $N^\bullet$ be a bounded above complex of finitely generated R-modules. If $N^\bullet \otimes^{\mathbf{L}}_R R/m = 0$ then $N^\bullet = 0$.*

Proof. Since N is bounded above, there exist an integer t such that $H^k(N^\bullet) = 0$ for all $k > t$. Since the tensor product is right exact , we have $H^t(N^\bullet \otimes^{\mathbf{L}}_R R/m) = H^t(N^\bullet) \otimes_R R/m = 0$ by the hypothesis. We have $H^t(N^\bullet) \otimes_R R/m = H^t(N^\bullet)/mH^t(N^\bullet) = 0$, since $H^t(N^\bullet)$ is a finitely generated R-module, by the usual Nakayama lemma 2.2.25 we get $H^t(N^\bullet) = 0$. Now we repeat the same argument by the assumption that $H^k(N^\bullet) = 0$ for all $k > (t-1)$. Therefore, by induction, $H^k(N^\bullet) = 0$ for all k. Hence $N^\bullet$ is acyclic, so $N^\bullet = 0$ in the derived category.

□

8.2 The proof

Finally we present the proof of the special case of the theorem 8.0.1 with assumptions: R is a field, $X \to Speck$ is a smooth projective morphism, $\mathcal{X}$ is a locally ringed space, and the map c is flat and bijective on closed points.

Proof. First we prove a general result with the conditions in the theorem 8.0.1.

Lemma 8.2.1. *Assume te conditions of 8.0.1 for the morphism of locally ringed spaces $c : \mathcal{X} \to X$. Let $y \in X$ be a closed point. There is a unique closed point $x \in \mathcal{X}$ such that $c^*\kappa(y) \simeq \kappa(x)$ in $D^b_{Coh}(\mathcal{X})$.*

Proof. First we prove the induced morphism on residue fields $\kappa(y) \to k(x)$ is an isomorphism. By condition (3) there is a unique closed point $x \in \mathcal{X}$ such that $c(x) = y$. By condition (5) we have the isomorphism $\mathcal{O}_{X,c(x)} \to \mathcal{O}_{\mathcal{X},x}$ is flat and $\kappa(c(x)) \to \kappa(c(x)) \otimes_{\mathcal{O}_{X,c(x)}} \mathcal{O}_{\mathcal{X},x}$ is an isomorphism. But we have the surjective morphism $\kappa(c(x)) \otimes_{\mathcal{O}_{X,c(x)}} \mathcal{O}_{\mathcal{X},x} \to \kappa(x)$. hence the composition $\kappa(c(x)) \to k(x)$ is a surjective field morphism, hence an isomorphism. Next we show the natural sheaf morphism $c^*\kappa(y) \simeq \kappa(x)$ is an isomorphism. For $t \in \mathcal{X}$ we have $(c^*\kappa(c(x)))_t = \kappa(c(x))_{c(t)} \otimes \mathcal{O}_{X,c(t)} \mathcal{O}_{X,t}$. By condition (3), c is bijective on closed points, so this is 0 if $t \neq x$ and for $t = x$ $\kappa(c(x)) \otimes \mathcal{O}_{X,c(x)} \mathcal{O}_{X,x} \cong \kappa(x)$. Therefore , the stalks of $\kappa(x)$ and $c^*\kappa(c(x))$ are the same. $c^*\kappa(c(x))$ is the sheafification of the presheaf C : $\mathcal{U} \mapsto \lim_{W \supseteq c(\mathcal{U})} \left(\kappa(c(x))(W) \otimes_{\mathcal{O}_X(W)} \mathcal{O}_X(\mathcal{U})\right)$. It is clear that $\mathcal{C}(\mathcal{U}) = 0$ if $x \notin \mathcal{U}$. It follows that $c^*\kappa(c(x))$ is isomorphic to the skyscrapper sheaf $\kappa(x)$. □

Let $D^b_{Coh}(X)$ (respectively $D^b_{Coh}(\mathcal{X})$) be the derived category of bounded complexes of $\mathcal{O}_X$-modules ($\mathcal{O}_{\mathcal{X}}$-modules) with coherent cohomology sheaves.

The morphism $c : \mathcal{X} \to X$ is flat, so it induces a derived pullback functor between bounded complex

$$c^* : D^b_{Coh}(X) \to D^b_{Coh}(\mathcal{X})$$

The proof can be divided in 5 steps:

Step 1: The functor c^* admits a right adjoint $c_* : D^b_{Coh}(\mathcal{X}) \to D^b_{Coh}(X)$.

We prove this using the represenatability theorem 8.1.26. Choose an arbitrary object $\mathcal{M} \in \mathrm{D}^b_{\mathrm{Coh}}(\mathcal{X})$. Define the cohomological functor

$$F_{\mathcal{M}} : \mathrm{D}^b_{\mathrm{Coh}}(X)^\circ \to \mathrm{Vect}(k) : P \mapsto \mathrm{Hom}_{\mathcal{O}_x}(c^*P, \mathcal{M})$$

X is smooth, therefore, all local rings are regular. By theorem 8.1.23, P is a perfect complex. Since the pullback preserves perfectness, c^*P is a perfect complex. We have

$$F_{\mathcal{M}}(P) = \mathrm{Hom}_{D(\mathcal{O}_{\mathcal{X}})}(c^*P, \mathcal{M}) \simeq \mathcal{H}^0\left(\mathrm{R}\Gamma\left(\mathcal{X}, c^*\left(P^\vee\right) \otimes^{\mathrm{L}}_{\mathcal{O}_{\mathcal{X}}} \mathcal{M}\right)\right)$$

with the dual object $P^\vee = \mathrm{R}\, Hom \mathcal{O}_X(P, \mathcal{O}_X)$. The second equality comes from the lemma 8.1.24 and c^*P being perfect. In addition, $c^*(P^\vee)$ is also a perfect complex and $c^*(P^\vee) \otimes^{\mathrm{L}}_{\mathcal{O}_x} \mathcal{M} \in \mathrm{D}^b_{\mathrm{Coh}}(\mathcal{X})$. It now follows from condition (2) that $\oplus_i F_{\mathcal{M}}(P[i])$ is a finite-dimensional k-vector space. This means that the cohomological functor $F_{\mathcal{M}}$ is of finite type.

Representability theorem 8.1.26 now implies that $F_{\mathcal{M}}$ is representable, i.e, there is an $M \in \mathrm{D}^b_{\mathrm{Coh}}(X)$ such that $F_{\mathcal{M}}(P) \simeq \mathrm{Hom}_{\mathcal{O}_X}(P, M)$ for all $P \in \mathrm{D}^b_{\mathrm{Coh}}(X)$.

Now define the functor $c_* : \mathrm{D}^b_{\mathrm{Coh}}(\mathcal{X}) \to \mathrm{D}^b_{\mathrm{Coh}}(X)$ by

$$\mathcal{M} \mapsto M$$

By the definition of the map $F_{\mathcal{M}}$, it is clear that c_* is the right adjoint of c^*.

Step 2:

Lemma 8.2.2. *For $P \in D^b_{Coh}(X)$ and $\mathcal{M} \in D^b_{Coh}(\mathcal{X})$, there is a projection formula*

$$(c_*\mathcal{M}) \otimes^L_{\mathcal{O}_X} P \simeq c_* \left(\mathcal{M} \otimes^L_{\mathcal{O}_x} c^* P\right)$$

Proof. X is smooth, so all local rings are regular and hence every object $\mathrm{D}^b_{\mathrm{Coh}}(X)$ is a perfect complex and is dualizable. Hence P is perfect. A similar argument to the proof of lemma 8.1.29 using the adjointness of c_* and c^* yields the result. □

Step 3: c^* is fully faithful.

By the adjointness of c^*, c_* we have $Hom_{\mathcal{O}_{\mathcal{X}}}(c^*P, c^*N) = Hom_{\mathcal{O}_X}(P, c_*c^*N)$ for $P, N \in\in D^b_{Coh}(X)$. Therefore, For the full faithfulness of c^*, it is sufficient to prove that $N \equiv c_*c^*N$ in the derived category $D^b_{Coh}(X)$, i.e, the natural map $\eta_N : N \to c_*c^*N$ is a quasi-isomorphism. By theorem 8.1.4, we should prove the cone of η_N, which we call H_N, is acyclic (zero in the derived category). It suffices to show the the local rings of H_N at closed points of X are zero. By the derived Nakayama lemma 2.2.25 it is sufficient to prove that $H_N \otimes^L_{\mathcal{O}_X} \kappa(y) \simeq 0$ for all closed points $y \in X$. Mapping cone commutes with taking derived tensor products, therefore

$$H_N \otimes^L_{\mathcal{O}_X} \kappa(y) = \text{cone } (N \to c_*c^*N) \otimes^L_{\mathcal{O}_X} \kappa(y) = \text{cone } (N \otimes^L_{\mathcal{O}_X} \kappa(y) \ \to c_*c^*N \otimes^L_{\mathcal{O}_X} \kappa(y))$$

By the projection formula 8.2.2, we have

$$c_*c^*N \otimes^L_{\mathcal{O}_X} \kappa(y) \simeq c_*(c^*N \otimes^L_{\mathcal{O}_{\mathcal{X}}} c^*\kappa(y)) = c_*c^*(N \otimes^L_{\mathcal{O}_X} \kappa(y))$$

since the derived tensor product commutes with the derived inverse image. Now putting all things together, we get

$$H_N \otimes^L_{\mathcal{O}_X} \kappa(y) = \text{cone } (N \otimes^L_{\mathcal{O}_X} \kappa(y) \ \to c_*c^*(N \otimes^L_{\mathcal{O}_X} \kappa(y))) = H_{N \otimes^L_{\mathcal{O}_X} \kappa(y)}$$

Therefore, $H_N \otimes^L_{\mathcal{O}_X} \kappa(y) \simeq 0$ is equivalent to that $H_{N \otimes^L_{\mathcal{O}_X} \kappa(y)}$ being acyclic or equivalently $\eta_{N \otimes^L_{\mathcal{O}_X} \kappa(y)}$ being a quasi-isomorphism. However, N $\otimes^L_{\mathcal{O}_X} \kappa(y)$ is the direct sum of shifts of $\kappa(y)$. Hence, we are reduced to proving that $\eta_{\kappa(y)} : \kappa(y) \to c_*c^*\kappa(y)$ is a quasi-isomorphism. By the lemma 8.2.1, there is a unique closed point $x \in \mathcal{X}$ with $c^*\kappa(y) \equiv \kappa(x)$. So, we should prove that $\kappa(y) \to c_*\kappa(x)$ is a quasi-isomorphism.

We want to prove that $c_*\kappa(x)$ is supported only at y. It suffices to prove that for $y' \neq y$ we have $(c_*\kappa(x)) \otimes^L_{\mathcal{O}_X} \kappa(y') = 0$. By theorem 8.2.1, let $x' \neq x \in \mathcal{X}$ be such that $c^*\kappa(y') \equiv \kappa(x')$. We have

$$(c_*\kappa(x)) \otimes^L_{\mathcal{O}_X} \kappa(y') \simeq c_* \left(\kappa(x) \otimes^L_{\mathcal{O}_x} \kappa(x')\right) \simeq 0$$

since $\kappa(x) \otimes^L_{\mathcal{O}_X} \kappa(x') = 0$.

Now that $c_*\kappa(x)$ is supported only at y, it follows that $c_*\kappa(x)$ is determined by its global sections $\mathrm{R}\Gamma(X, c_*\kappa(x))$. We have

$$\mathrm{R}\Gamma(X, c_*\kappa(x)) = \mathrm{RHom}_{\mathcal{O}_X}(\mathcal{O}_X, c_*\kappa(x)) \simeq \mathrm{RHom}_{\mathcal{O}_x}(\mathcal{O}_x, \kappa(x)) \simeq \mathrm{R}\Gamma(\mathcal{X}, \kappa(x)) \simeq \kappa(x)$$

Therefore, $c_*\kappa(x)$ is isomorphic to $\kappa(x)$. Moreover, under the morphism $c^* : \kappa(y) \to \kappa(x)$, $\kappa(y)$ is isomorphic to $\kappa(x)$. Hence $\kappa(y) \to c_*\kappa(x)$ is a quasi-isomorphism.

Step 4: c^* is essentially surjective.

We prove this in a similar way to the previous step. For an arbitrary object $\mathcal{M} \in \mathrm{D}^b_{\mathrm{Coh}}(\mathcal{X})$, we should prove there is a $M \in \mathrm{D}^b_{\mathrm{Coh}}(X)$ with $c^*M \simeq \mathcal{M}$. We show that the obvious choice $M = c_*\mathcal{M}$ works. Hence we should prove that the adjunction $\epsilon_{\mathcal{M}} : c^*c_*\mathcal{M} \to \mathcal{M}$ is a quasi-isomorphism. We proceed like before. We must show the cone of $\epsilon_{\mathcal{M}}$, namely $E_{\mathcal{M}}$, is acyclic. Again by the derived Nakayama lemma 8.1.30, it suffices to prove that $E_{\mathcal{M}} \otimes^{\mathrm{L}}_{\mathcal{O}_x} \kappa(x) \simeq 0$ for all closed points $x \in \mathcal{X}$.

By the lemma 8.2.1, there is a unique closed point $x' \in X$ with $\kappa(x) \simeq c^*\kappa(x')$. So, it is sufficient to prove that $E_{\mathcal{M}} \otimes^{\mathrm{L}}_{\mathcal{O}_x} c^*\kappa(x') \simeq 0$ for all closed points x' of X.

By the projection formula 8.2.2 and a similar argument as in step 3, we have

$$E_{\mathcal{M}} \otimes^{L}_{\mathcal{O}_x} c^*\kappa\left(x'\right) \simeq E_{\mathcal{M} \otimes^{L}_{\mathcal{O}_x} c^*\kappa(x')}.$$

$\mathcal{M} \otimes^{\mathrm{L}}_{\mathcal{O}_x} c^*\kappa(x')$ is a finite direct sum of shifts of $c^*\kappa(x')$. Therefore, it suffices to prove that $E_{c^*\kappa(x')}$ is acylic or equivalently $\epsilon_{c^*\kappa(x')} : c^*c_*c^*\kappa(x') \to c^*\kappa(x')$ is a quasi-isomorphism. But, in step 3, we showed that $\kappa(x') \to c_*c^*\kappa(x')$ is a quasi-isomorphism.

Step 5: we have the cohomological comparison results, $\mathrm{H}^i(X, F) \to \mathrm{H}^i(\mathcal{X}, c^*F)$.

Let $F \in \mathrm{Coh}(X)$. Note that the functor $c^* : \mathrm{D}^b_{\mathrm{Coh}}(X) \to \mathrm{D}^b_{\mathrm{Coh}}(X)$ is fully faithful. Hence we have

$$\begin{aligned} \mathrm{H}^i(X, F) &\simeq \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, F[i]) \simeq \mathrm{Hom}_{\mathcal{O}_x}(c^*\mathcal{O}_X, c^*F[i]) \\ &\simeq \mathrm{Hom}_{\mathcal{O}_x}(\mathcal{O}_x, c^*F[i]) \simeq \mathrm{H}^i(X, c^*F) \end{aligned}$$

Therefore $\mathrm{H}^i(X, F) = \mathrm{H}^i(X, c^*F)$ for all $i \geq 0$.

□

9 References

References

[1] J.P. Serre, *Faisceaux algt'ebriques coht'erents*, Ann. of Math. 61 (1955), 197278.

[2] J.P. Serre, *Géométrie algébrique et géométrie analytique*, Université de Grenoble. Annales de lInstitut Fourier 6: 142, (1956)

[3] A. Grothendieck, *Sur quelques points d'algèbre homologique*, Tôhoku Mathematical Journal, (2), 9 (2): 119221, 1957.

[4] Grothendieck, Alexandre; Dieudonné, Jean, *Éléments de géométrie algébrique: II*, Publications Mathématiques de l'IHÉS.

[5] Grothendieck, Alexandre; Dieudonné, Jean, *Éléments de géométrie algébrique: III*, Publications Mathématiques de l'IHÉS.

[6] R. Hartshorne, *Algebraic Geometry*, Springer, 1977.

[7] *Stacks Project*, URl:`https://stacks.math.columbia.edu`, 2021.

[8] O. Zariski, P. Samuel, *Commutative algebra* , 2 , Springer, 1975.

[9] D. Eisenbud *Commutative Algebra: With a View Toward Algebraic Geometry*, Springer, 1995.

[10] M. Atiyah and I. G. Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley, 1969.

[11] R. Michael Range, *Complex Analysis: A Brief Tour into Higher Dimensions*, The Mathematical Association of America [Monthly 110, February 2003].

[12] K. Kedlaya, *Algebraic Geometry lecture notes*, URL: `https://ocw.mit.edu/courses/mathematics/18-726-algebraic-geometry-spring-2009/lecture-notes`

[13] J. Harris, P. Griffiths, *Principles of Algebraic Geometry*, Wiley Classics Library, 1978.

[14] H. Grauert, R. Remmert, *Coherent analytic sheaves*, Springer, 1984.

[15] R. Gunning, H. Rossi, *Analytic functions of several complex variables*, Prentice-Hall, 1965.

[16] Daniel Huybrechts, *Complex Geometry: An Introduction*, Springer, 2005.

[17] Illusie L., *Grothendieck's existence theorem in formal geometry(with a letter of Jean-Pierre Serre)*, Fundemental Algebraic Geomtery Grothendieck's FGA Explained, American Mathematical Society, 2005.

[18] Doan Trung Cuong, *A minicourse on Formal Geometry*, URL: `https://www.uni-due.de/~mat903/sem/ws0809/material/Minicourse_FormalGeometry.pdf`

[19] A. Neeman, *Algebraic and Analytic Geometry*, Cambridge University Press, 2007.

[20] R. Vakil, *Foundations of Algebraic Geometry*, URL: `http://math.stanford.edu/~vakil/216blog/FOAGnov1817public.pdf`

[21] Daniel Huybrechts, *Fourier-Mukai transforms in algebraic geometry*, Clarendon Press 2006.

[22] A. I. Bondal and M. M. Kapranov, *Representable functors, Serre functors, and mutations*, Izv. Akad. Nauk SSSR Ser. Mat. 53 (1989), no. 6, 11831205, 1337.

[23] A. Bondal and M. Van Den Bergh, *Generators and representability of functors in commutative and noncommutative geometry*, Mosc. Math. J. 3 (2003), no. 1, 136, 258.

[24] Jack Hall, *GAGA theorems*, URL: `https://arxiv.org/abs/1804.01976v2`

[25] Amnon Neeman, *Triangulated categories with a single compact generator and a Brown representability theorem*, URL: `https://arxiv.org/abs/1804.02240v4`