Generalized Hodge Index Conjecture for Singular Algebraic Varieties

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Signature

Given a connected and oriented closed real manifold X of dimension $4k$, cup product gives rise to a bilinear form Q on the 'middle' real cohomology group

$$Q : H^{2k}(X,\mathbb{R}) \otimes H^{2k}(X,\mathbb{R}) \to \mathbb{R}$$

This form is symmetric and nondegenerate (Poincaré duality)

$$\sigma(X) := \sigma(Q) = n^+ - n^-$$

$n^{\pm}$: $\# \pm$ entries in any diagonal matrix defining Q.

S

- Thom: The signature is a bordism invariant, $\sigma : \Omega^{SO} \otimes \mathbb{Q} \to \mathbb{Q}$
- Pontryagin classes are complete invariant of $\Omega^{SO} \otimes \mathbb{Q} = \mathbb{Q}[\mathbb{CP}^2, \mathbb{CP}^4, \ldots]$.

Hirzebruch Signature Theorem : (1978)

$$\sigma(X) = \langle (L_n(p_1(T_X), \ldots, p_n(T_X)), [X] \rangle = L_0(X)$$

Hirzebruch L classes

$$Q_1(x) = \frac{x}{tanh\, x} = 1 + \frac{x^2}{3} - \frac{x^4}{45} + \dots$$

$$\prod_{i=1}^{n} Q_1(x_i) = 1 + L_1(p_1) + L_2(p_1, p_2) + \dots + L_n(p_1, \dots, p_n) + L_{n+1}(p_1, \dots, p_n, 0) + \dots$$

$$p_j = \sigma_j(x_1^2, \dots, x_n^2)$$

$$L_r \in \mathbb{Q}[p_1, \dots, p_r]$$

$$L^i(X) := L_i(p_1(T_X), \dots, p_n(T_X)) \in H^{4i}(X, \mathbb{Q})$$

Theorem: (Browder–Novikov 1972)

The homotopy type and L-classes determine the diffeomorphism class of a smooth simply connected manifold of dimension ≥ 5 up to a finite number of possibilities.

Another construction of L classes

$$f : M^n \to S^{n-4i}$$

$p \in S^{n-4i}$ a regular value, $f^{-1}(p) \hookrightarrow M$ smooth closed submanifold.

$$f \mapsto \sigma(f^{-1}(p))$$

Bordsim invariance of signature depends only on homotopy class of f.

$$\pi^{n-4i}(M) \xrightarrow{\sigma} \mathbb{Z}$$

$\pi^{n-4i}(M) = [M, S^{n-4i}]$ is a group for $4i < \frac{n-1}{2}$

$$\text{Hurewicz: } \pi^k(M) \to H^k(M, \mathbb{Z}), f \mapsto f^*(u), < u, [S^k] = 1 >$$

$$Serre : \pi^k(M) \otimes \mathbb{Q} \to H^k(M, \mathbb{Q})$$

isomorphism, $n < 2k - 1.$ $\to 4i < \frac{n-1}{2}$, *leads to* $\quad l_{n-4i}(M) = \quad H^{n-4i}(M, \mathbb{Q}) \xrightarrow{\sigma \otimes \mathbb{Q}} \mathbb{Q}$

$$l_{n-4i}(M) = L^i(M) \cap [M]$$

Hodge index theorem

$X =$ A compact complex manifold of complex dimention $2n$. The underlying real manifold is $4n$ dimentional.

$$h^{p,q}(X) := dim_{\mathbb{C}}[H^{p,q}(X) \cong H^q(X, \Omega^p)]$$

Hodge Index theorem

$$L_0(X) = \sigma(X) = \sum_{p,q} (-1)^p \, h^{p,q}(X) = \chi_1(X)$$

If X is Kahler, it follows from polarization of Hodge strcutres.
For a general compact complex manifold, this can be proved using arguments based on Atiyah–Singer index theorem 1963.

Hodge structures, X = smooth projective complex algebraic variety

- Degeneration of Hodge–de Rham spectral sequence at E_1:
$$H^q(X,\Omega^p_X) \Rightarrow H^{p+q}(X,\Omega^\bullet_X) \overset{P.L.}{=} H^{p+q}(X,\mathbb{C})$$
-
$$H^r(X,\mathbb{C}) = \bigoplus_{p+q=r} H^{p,q}(X), \qquad H^{p,q}(X) = gr^p_F H^{p+q}(X,\mathbb{C}) = H^q(X,\Omega^p_X)$$
-
$$\text{Lefschetz: } L^{n-k} : \Omega^p_{X,\mathbb{R}} \to \Omega^{2n-k}_{X,\mathbb{R}}$$
-
$$Q(a,b) : H^k(X,\mathbb{R}) \otimes H^k(X,\mathbb{R}) \to \mathbb{R} \qquad \langle L^{n-k}a, b\rangle = \int_X w^{n-k} \wedge a \wedge b$$
$$H_k(a,b) := i^k Q(a,\overline{b}), \quad \text{a Hermitian form on } H^k(X,\mathbb{C})$$
-
$$\text{Lefschetz Primitive Decompostion: } H^k(X,\mathbb{C}) = \bigoplus_{2r\le k} L^r H^{k-2r}_{prim}(X,\mathbb{C})$$
- Polarization (Riemann bilinear relation): $H^{p,q}$ form an orthogonal direct sum for H_k .The form $(-1)^{\frac{k(k-1)}{2}} i^{p-q-k} H_k$ is positive definite on the complex subspace $H^{p,q}$.

Hodge index theorem

X = A compact smooth complex algebraic variety of dimention $2n$.

$$h^{p,q}(X) = dim_{\mathbb{C}}[H^{p,q}] = dim_{\mathbb{C}}[gr_F^p H^{p+q}(X,\mathbb{C})]$$

Hodge Index theorem

$$\sigma(X) = \sum_{i,p}(-1)^p dim[gr_F^p H^\bullet(X,\mathbb{C})] = \chi_1([H^\bullet(X)])$$

Immediate from the sign of H_k on primitive parts and $sign(Q : H^{2n}(\mathbb{R})) = sign(H_k : H^n(\mathbb{C}))$

Hirzebruch characteristic classes

X is a smooth compact complex algebraic variety.

$$Q_y(x) = \frac{x(1+y)}{1-e^{-x(1+y)}} - xy \in \mathbb{Q}[y][[x]]$$

$$T_y^*(T_X) := \prod_i Q_y(a_i) \in H^\bullet(X, \mathbb{Q})$$

where $\{\alpha_i\}$ are the Chern roots of the tangent bundle T_X. Note that $Q_y(\alpha)$ is equal to

$$Q_y(x) = \begin{cases} 1+x & \text{for } y=-1 \\ \frac{\alpha}{1-e^{-x}} & \text{for } y=0 \\ \frac{x}{\tanh x} & \text{for } y=1 \end{cases}$$

$$T_y^*(T_X) = \begin{cases} c^*(T_X) & \text{for } y=-1 \\ \mathrm{td}^*(T_X) & \text{for } y=0 \\ L^*(T_X) & \text{for } y=1 \end{cases}$$

Generalized Hirzebruch-Riemann-Roch theorem

Hirzebruch genus: X : A compact complex manifold.

$$\chi_y(X) := \chi_y(X, \mathscr{O}_X) := \sum_{p \geq 0} \chi(X, \Omega_X^p) \cdot y^p = \sum_{p \geq 0} \left(\sum_{i \geq 0} (-1)^i \dim H^i(X, \Omega_X^p) \right) \cdot y^p$$
$$= \sum_{p,q} (-1)^q \, h^{p,q}(X) y^p$$

Hirzebruch-Riemann-Roch theorem

$$\chi_y(X) = \langle T_y^*(T_X), [X] \rangle$$

Special cases

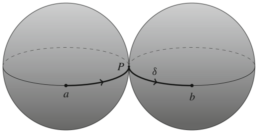
$$\chi_y(X) = \begin{cases} \langle c^*(T_X), [X] \rangle = \chi(X) & y = -1 \quad (\textit{Gauss–Bonnet–Chern}) \\ \langle \mathrm{td}^*(T_X), [X] \rangle = \chi_a(X) & y = 0 \quad (\textit{Riemann} - \textit{Roch}) \\ \langle L^*(T_X), [X] \rangle = \sigma(X) & y = 1 \quad (\textit{Hirzebruch signature}) \end{cases}$$

What if X is not smooth?

- (Co)homology theory? Poincare duality fails for singular cohomology of singular spaces.

$$X = \{z_0 z_1 = 0\} \cong \mathbb{C}P^1 \cup \mathbb{C}P^1 \subset \mathbb{C}P^2$$

$$H^0 = \mathbb{C} = \langle a \rangle = \langle b \rangle, H^2 = \mathbb{C}^2$$

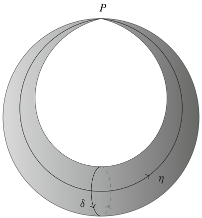


X

- Hodge theory? Hodge-Lefchetz theory fails for singular cohomology of singular algebraic spaces.

$$Y = \{z_0^3 + z_1^3 = z_1 z_2 z_3\} \subset \mathbb{C}P^2$$

$$H^1(Y) = \mathbb{C} = \langle \eta \rangle$$



Y

- Characteristic classes? How to define them? There is no tangent space?

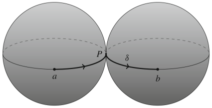
Intersection Homology-Middle perversity

Mark Goresky and Robert MacPherson 1974.

$$IH_i(X) := H_i(IC^m_\bullet(X))$$

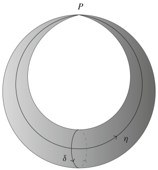
$IC^m_\bullet(X) \subset C_\bullet(X)$ singular chains such that both the chain and its boundary are "allowable" singular simplexes. "allowable": alowable deviation from transversality to a stratification

- 1-cycles are not allowed to pass through P. δ not allowed, 2-cycles are allowed.



$$IH^0 = \mathbb{C}^2 = \langle a \rangle \oplus \langle b \rangle, \quad IH^2(X) = \mathbb{C}^2$$

- 1-cycles are not allowed to pass through P, but 2-cycles are allowed. So $[\eta]$ is not allowed



$$IH^1(X) = 0$$

Poincare duality and signature, X compact, $dim = 4n$

$$IH_{2n}(X, \mathbb{Q}) \otimes IH_{2n}(X, \mathbb{Q}) \to \mathbb{Q}$$

Sheaf theoretic IC complex

X : n dimensional pseudomanifold middle

Deligne intersection cohomology complex

Stratifcation

$$X = X_n \supseteq X_{n-2} \supseteq X_{n-3} \supseteq \ldots \supseteq X_0 \supseteq \emptyset.$$

$$j_k : U_k \hookrightarrow U_{k+1} \quad , U_k = X - X_{n-k}, \quad k \geq 2$$

$$IC_X^\bullet(\mathbb{Q})|_{U_2} = \mathbb{Q}_{U_2}[n].$$

$$IC_X^\bullet(\mathbb{Q})|_{U_{k+1}} := \tau_{\leq \frac{(k-2)}{2} - n} Rj_{k_*} \left(IC_X^\bullet(\mathbb{Q})|_{U_k} \right)$$

Goresky-MacPherson 1983

$$IH_\bullet(X, \mathbb{Q}) = \mathbb{H}_c^{-\bullet}(X, IC_X^\bullet(\mathbb{Q}))$$

Perverse sheaves, X : A complex algebriac variete $dim_{\mathbb{C}} = n$

$IC^{\bullet}(L)$ is a perverse sheaf

$$\mathcal{F}^{\bullet} \in {}^{p}D^{\leq 0}(X) \Longleftrightarrow dim_{\mathbb{C}} Supp^{-j}(\mathcal{F}^{\bullet}) \leq j, \forall j \in \mathbb{Z},$$

$$\mathcal{F}^{\bullet} \in {}^{p}D^{\geq 0}(X) \Longleftrightarrow dim_{\mathbb{C}} Cosupp^{j}(\mathcal{F}^{\bullet}) \leq j, \forall j \in \mathbb{Z}$$

Defines the Perverse $t-$structure τ on $D^{b}_{cons}(\mathbb{C}_X)$ with $Perv(X) = \heartsuit(\tau)$.

Intermediate extention

$j : U \hookrightarrow X$ be an open inclusion, $G \in D^b_c(U)$, $F \in D^b_c(X)$ any extension ($j^* F = G$)

$$j_! G \to F \to Rj_* G \qquad j_{!*}(G) := im(pj_! G \to pRj_* G)$$

U the open smooth locus: $IC_X(L) = j_{!*}(L[n])$

Generalized Poincare duality

$$D_X(IC_X(\mathbb{Q})) \cong IC_X(\mathbb{Q})$$

Hodge structures

Deligne 1971

Singular cohomology $H^\bullet(X, \mathbb{Z})$ of any complex algebraic variety admits a mixed Hodge structure.

Questions

- Does $H^\bullet(X, \mathcal{L})$, for a local system $\mathcal{L}$ on X admit mixed Hodge structure?
- How to put Hodge structures on $IH^\bullet(X) = H^\bullet(X, IC_X^\bullet(\mathbb{Q}))$?
- Goresky-MacPherson conjecture: for X projective, there should be a Kahler package results (Hodge structres-Hyperplane, Hard Lefchetz theorems) for $IH^\bullet(X)$.

Saito 1988:

$H^\bullet(X, IC_X^\bullet(L))$ has mixed Hodge structures and satisfies Kahler-Package, for L a polarizable variation of Hodge structures.

Motivation from the arithmetic world

Deligne's Analogy ICM 1970

$$\{Hodge\ theory\ on\ X/\mathbb{C}\} \Leftrightarrow \{Galois\ rep.\ on\ l{-}adic\ cohomology\ on\ X/\mathbb{F}_q\}$$

$$\{Pure\ weight\ w\ (\ H^i\ )\} \Leftrightarrow \{Purity\ (eigenvalues\ of\ Frobenius\ on\ H^i_{et}\ have\ absolute\ value\ q^{(i+w)/2})\}$$

BBD(G) 1982

X Projective. U open smooth, $j : U \hookrightarrow X, IC_X(\mathbb{Q}_l) := j_{!*}(\mathbb{Q}_l[n])$

- Weil conjectures: $IC_X(\mathbb{Q}_l)$ is pure.
- Hard Lefschetz: $\eta : c_1$ of an ample invertible sheaf, the cup product is an isomorphism:

$$\eta^i : H^{N-i}\left((X/_{\overline{F_q}})_{et}, IC_X(\mathbb{Q}_l)\right) \to H^{N+i}\left((X/_{\overline{F_q}})_{et}, IC_X(\mathbb{Q}_l)\right)$$

Corollary for complex projective variety

The hard Lefschetz (and the decomposition theorem) hold for intersection cohomology of $X/\mathbb{C}$.

Riemann-Hilbert correspondence

(Kashiwara 1980-84)

X a smooth complex algebraic variety, there is an equivalence from the categories

$$DR : D^b_{r,h}(X) \to D^b_{cons}(X^{an}, \mathbb{C})$$

$$\mathcal{M} \mapsto [\Omega_{X^{an}} \otimes^L_{D_X} (\mathcal{M}^{an})]$$

Heart: $\{$*Regular holomomic (algebraic) D-modules on* $X\} \Longleftrightarrow \{$*Perverse sheaves on* $X^{an}\}$

$$O_X \to [\Omega^1_{X^{an}} \to \Omega^2_{X^{an}} \to \ldots \to \Omega^n_{X^{an}}] \overset{\text{q.iso}}{=} \mathbb{C}_{X^{an}}$$

Saito's theory of mixed Hodge modules

$X =$ a complex algebraic variety. Abelian category $\mathrm{MHM}(X)$. If X smooth, $((\mathcal{M}, F), K, W)$:

- $(\mathcal{M}, F)$: Algebraic (regular) holonomic $\mathcal{D}$-module with a good filtration F, i.e., by a bounded from below, increasing, and exhaustive filtration $F_p\mathcal{M}$ by coherent algebraic $\mathcal{O}_X$-modules such that

$$F_i\mathcal{D}_X\left(F_p\mathcal{M}\right) \subset F_{p+i}\mathcal{M}$$

- $K \in Perv(X^{an}, \mathbb{Q})$. Both K, M endowed a finite increasing filtration W.
- An isomorphism α compatible with W under the Riemann-Hilbert correspondence

$$\alpha : \mathrm{DR}(\mathcal{M})^{an} \simeq K \otimes_{\mathbb{Q}_X} \mathbb{C}_X$$

- "Polarizable Hodge modules: $S : M \otimes M \to D_X(-w)$, $M \cong D_X(M)(-w)$

MHM over a point

$$MHM(pt) = mHs^p = \textit{polarizable rational mixed Hodge structures.}$$

Properties

Forgetful functor *rat*

$$rat : MHM(X) \to Perv(\mathbb{Q}_X) \qquad ((\mathcal{M}, F), K, W) \mapsto K$$

is exact and faithful. Six-functor formalism, $f^*, f_*, f_!, f^!, \otimes, \boxtimes$ on $D^b MHM(X)$.

grDR

$$gr^F \mathrm{DR}_y : K_0(\mathrm{MHM}(X)) \to K_0(X)\left[y, y^{-1}\right]$$

If X is smooth and $M \in \mathrm{MHM}(X)$, $(M, F_\bullet M)$ the underlying filtered D_X-module of M

$$gr^F \mathrm{DR}_y[M] := \sum_p \left[\mathrm{Gr}^F_{-p} \mathrm{DR}(M)\right] \cdot (-y)^p = \sum_{p,i} (-1)^i \left[H^i \, \mathrm{Gr}^F_{-p} \mathrm{DR}(M)\right] \cdot (-y)^p$$

X smooth, pure dimentional, algebraic (holomorphic) p-forms

$$gr^F_{-p} DR(\mathbb{Q}^H_X) = \Omega^p_X[-p]$$

Examples

Constant Hodge module

$$a_X : X \to pt$$

Constant Hodge module $\mathbb{Q}_X^H$:

$$\mathbb{Q}_X^H := a_X^* \mathbb{Q}^{Hdg},\ \mathbb{Q}^{Hdg} = \mathbb{Q}(0) \in MHM(pt) = MHS^p$$

$$(a_X)_* \mathbb{Q}^{Hdg} = H_\bullet(X, \mathbb{Q}) \in mHs^p$$

$$rat(\mathbb{Q}_X^H) = \mathbb{Q}_X$$

Deep theorem of Saito: This coincides with the Deligne's mixed Hodge structures.

The underlying filtered D-module $(\mathcal{M}, F)$ is very complicated, if X is very singular.

IC module

$IC_X^H(L)$ module

- $L =$ "admissible" variation of polarized (pure) Hodge structures (of weight k) on a smooth open set $j : U \hookrightarrow X$, $(\mathcal{V}, \nabla) = L \otimes_{\mathbb{C}} O_X$
- $IC_X^H(L) \in MHM(X)$ (of pure weight $k+n$), $\quad rat(IC_X^H(L))| = IC_X(L) = j_{!*}(L[n])$
- Normal crossing case: $\mathcal{M}_{IC_X^H(L)} = D_X$-submodule of $j_*\mathcal{V} = \langle \tilde{\mathcal{V}} \rangle$, Deligne's canonical extention whose residues along the components D have eigenvalues in $[0,1)$.

Corrolary: Intersection cohomology

$$IC_X^H := IC_X^H(\mathbb{Q}_U)$$

$$(a_X)_*(IC_X^H) = IH^\bullet(X, \mathbb{Q}) \in mHs^p$$

$$IC_X \text{ is Verdier self-dual as a perverse sheaf} \to \text{Poincare duality} : D_X(IC_X^H) = IC_X^H(n)$$

$$Gr_n^W H^n \mathbb{Q}_X^H = IC_X^H$$

L class of compact algebraic variety X

$$\begin{tikzcd}
K_0(Var/X) \arrow[r, "\chi_{Hdg}"] \arrow[rr, bend left, "T_{1*}"] \arrow[dr, "sd"'] & K_0(\mathrm{MHM}(X)) \arrow[r, "MHT_{1*}"] & H_*(X, \mathbb{Q}) \\
& \Omega_{\mathbb{R}}(X) \arrow[ur, "L_*"'] &
\end{tikzcd}$$

- χ_{Hdg}: $[f : Y \to X] \mapsto f_!(\mathbb{Q}_Y^H)$
- MHT_{1*} : $K_0(\mathrm{MHM}(X)) \xrightarrow{gr^F DR_1} K_0(X) \xrightarrow{td_2} H_*(X, \mathbb{Q})$,
- td_* : $K_0(X) \to H_*(X, \mathbb{Q})$ Baum-Fulton-MacPherson Todd class transformation
- L_* : Homology L-class transformation (Cappell-Shaneson 1991):
- $\Omega_{\mathbb{R}}(X)$: cobordism group of selfdual $\mathbb{R}-$constructible complexes $S : \mathcal{F} \otimes \mathcal{F} \to D_X$
- sd : the unique selfduality transformation ([BSY] 2005) with normalization $sd(id_X) = [\mathbb{Q}_X[n]]$ for X smooth compact, pure n-dimensional.

Existence of sd comes from the nontrivial theorem of F. Bittner that $K_0(Var/X)$ can be obtained by blow up of proper morphisms of smooth proper varieties over X.

Problem

X not smooth, have the self dual complex of Intersection complex induced by Poincare Duality

$$[IC_X(\mathbb{R})] : IC_X(\mathbb{R}) \otimes IC_X(\mathbb{R}) \to D_X$$

$$L_*(X) := L_*(IC_X(\mathbb{R}))$$

For $IC'^H_X = IC^H_X(\mathbb{R})[-n]$, $rat(IC'^H_X) = IC_X(\mathbb{R})$

$$IT_{1*}(X) := MHT_{1*}(IC'^H_X)$$

$$T_{1*}(X) = T_{1*}(id_X) = MHT_{1*}(\mathbb{R}^H_X)$$

In general $T_{1*}(X) \neq IT_{1*}(X)$

Generalized Hodge index conjecture for intersection homology L-classes: [BSY 2005]

For any compact complex algebraic variety X, $\mathrm{IT}_{1*}(X) \overset{?}{=} L_*(X)$

Known results

Hodge index theorem for intersection cohomology, Saito 1988

$$L_0(X) = [IT_{1*}(X)]_0$$

$$\sigma\left(IH^n(X) \otimes IH^n(X) \to \mathbb{R}\right) = \chi_1(IH^n(X))$$

$$For\ X = pt,\ MHT_{1*}([V]) = \chi_1([V])$$

(Bobadilla, Pallares, Saito 2022)

The conjecture Holds for compact $\mathbb{Q}-$rational Homology manifold $(IC'_X(\mathbb{Q}) = \mathbb{Q}_X)$.

The conjecture Holds for compact toric and Schubert varieties.

Polarization and Pol functor

$$\begin{tikzcd} K_0(Var/X) \arrow[r, "\chi_{Hdg}"] \arrow[dr, "sd"'] & K_0(\mathrm{MHM}(X)) \arrow[r, "MHT_{1*}"] \arrow[d, dashed, red, "Pol"] & H_*(X, \mathbb{Q}) \\ & \Omega_{\mathbb{R}}(X) \arrow[ur, "L_*"'] & \end{tikzcd}$$

[BPS 2022]

There is a natural transformation

$$Pol : K_0((X)) \to \Omega_R(X)$$

$$Pol([M]) = [(F_R, (-1)^{w(w+1)/2} S_R)] \in \Omega_R(X),$$

for a pure Hodge module M of weight w on X, with $(F_{\mathbb{R}}, S_{\mathbb{R}})$ a polarization.

Warning

The left triangle is commutative but the right triangle is not known to be commutative