Lifting results from finite characteristic to $\mathbb{C}$

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1 Introduction

This is my translation of some ideas given in Section 6 of the BBD paper. There might be numerous errors. Use it at your own risk.

Deligne gave another proof of the Hard Lefschetz theorem for smooth complex projective varieties, which originally was proved by analytical techniques on a Kähler manifold, using the analogous l-adic Hard Lefschetz theorem, which he proved in routes of proving Weil conjectures. The decomposition theorem of Beilinson, Bernstein, and Deligne regarding proper pushforwards of perverse sheaves on complex varieties was also obtained from its l-adic counterpart.

The method of lifting results from finite characteristic and l-adic cohomology to results about complex algebraic varieties and singular cohomology is implicit in Deligne's papers, but illustrated more in the last section of the BBD paper.

2 Analytic topology and Etale topology

The first step is to talk about the bridge connecting etale topology and analytic topology. This is the good old analytification functor of Serre introduced in GAGA. In the language of sites and topoi, we have a map of sites

$$\epsilon : X(\mathbb{C}) \to X_{et}$$

Recall that this means there is a continuous map of etale covers to analytified etale covers. There is a pullback functor of sheaves

$$\epsilon^* : Sh(X_{et}) \to sh(X(\mathbb{C}))$$

defined by taking colimits of etale neighborhoods.

1. ϵ^* defines an equivalence of categories from constructible etale sheaves to constructible sheaves on $X(\mathbb{C})$. Here constructible means locally constant on strata of a stratification with finite stalks.

2. If R is a finite ring, ϵ^* defines an equivalence of categories from constructible sheaves of R-modules on X_{et} to constructible sheaves of R-modules on $X(\mathbb{C})$.

3. ϵ^* defines an equivalence of categories from l adic $\mathbb{Z}_l$ constructible sheaves on X_{et} to constructible sheaves of $\mathbb{Z}_l$ modules. Here to see why a sequence of $\{F_n^{an}\}$ of $\mathbb{Z}/l^n$ comes from a $\mathbb{Z}_l$ modules, we can use the fact that an analytic contractible neighborhood trivializes any trivialize $\mathbb{Z}/l^n$ module and complex varieties are locally contractible in analytic topology.

4. For $\mathbb{Q}_l$ sheaves, ϵ^* is fully faithful but not essentially surjective. For normal varietes a locally constant sheaf of $\mathbb{Q}_l$ modules is in the image iff $\pi_1(X(\mathbb{C}), x)$ stabilizes a $\mathbb{Z}_l$ lattice. The same is true for constructible sheaf if the above condition is satisfied for each normal strata.

5. For a constructible torsion sheaf on X_{et}, we have base change isomorphism

$$\epsilon^* R^q f_* F \to R^q f_*(\epsilon^* F)$$

for any morphism of finite type over $\mathbb{C}$, $f : X \to Y$.

3 Finitely generated over $\mathbb{Z}$ models

The goal is to observe that most of these algebraic objects, like spaces and sheves are some finite type objects. At the end for sheaves we reduced it mod maximal and they become torsion sheaves and we use part 5 of the previous section.

we can write

$$\mathbb{C} = \lim_{\rightarrow}\{\text{finitely generated rings over } \mathbb{Z}\}$$

Any finite type "object" over $\mathbb{C}$ is defined over some finitly generated ring over $\mathbb{Z}$ like R. If X is finite type over $\mathbb{C}$, then there is such a ring A. We use the notation X_A/A. Let $m \subset R$ be a maximal ideal. Then $k = R/m$ is a finite field with q elements.

4 Applications

4.1 Smooth projective case

The smooth base change theorem (for X smooth, projective) implies

$$H_{et}(X \otimes_R \overline{k}, Q_l) = H_{et}(X \otimes_R \mathbb{C}, Q_l) = (\lim_{\leftarrow} H_{et}(X \otimes_R \mathbb{C}, \mathbb{Z}/l^n)) \otimes Q_l = H_{an}(X(\mathbb{C})), Q) \otimes Q_l \tag{1}$$

The first two coefficients mean l-adic sheaves. The last equality comes from the vanishing of the first derived inverse limit by the Mittag-Leffler condition. The first equality is equivariant w.r.t the action of the geometric Frobenious F_m.

Use Weil conjectures for $H_{et}(X \otimes_R \overline{k}, Q_l)$.

4.2 Compatibility of Hodge theoretic and l-adic Weight filtrations

The compatibility goes back to Deligne in his ICM 1970 talk. Here I borrow the wording of A. Durfee in his "The l-adic cohomology of links" paper:

Define the Weight filtration on $H_{et}(X \otimes_R \mathbb{C}, Q_l)$ as:

$\overline{W}_s = \{v \in H|$**v is a generalized eigenvector of** F_m **with eigenvalue** $q^{s/2}\}$

Let W_k be the filtration induced by the weight filtration on the mixed Hodge structure of $H_{an}(X(\mathbb{C})), Q)$.

Theorem: We have $W_k = \bigoplus_{s \leq k} \overline{W}_s$, i.e., l-adic weight filtration splits the weight filtration of mixed Hodge structure.

Proof: Follows from the spectral sequence and the smooth case above.

4.3 Finitely generated fundamental groups

Assume X is finite type over $\mathbb{C}$. let F be a constructible sheaf of $\mathbb{C}$ vector spaces. $U \subset X$ a connected dense smooth open subvariety. Assume for $j : u \in U$, $\pi_1(U, u)$ is finitely generated. Assume moreover that

$$F = j_! L$$

for a local system L on U. Then L corresponds to a representation

$$\rho : \pi_1(U, u) \to GL(n, \mathbb{C})$$

Since $\pi_1(U, u)$ is finitly generated there is a finitely generated ring A over $\mathbb{Z}$, where $L = L_A \otimes \mathbb{C}$, where L is a local system of A modules. Let $F_A = j_! L_A$.

Assume for any finite type A-module M, we have isomorphism

$$R_{an}\Gamma(X, F_A \otimes M) = R_{an}\Gamma(X, F_A) \otimes M$$

In particular,

$$H(X, F_A \otimes M) = H(X, F_A) \otimes M$$

Let $F_k = F_A \otimes A/m$ for a maximal ideal m. Since k is finite, F_k is in the image of ϵ^*, i.e., there is a sheaf F_k^{et} on X_{et} corresponding to it.

So,

$$H_{an}(X, F_A) \otimes k = H_{an}(X, F_k) = H_{et}(X_{et}, F_k^{et})$$

The first equality comes from the assumption above and the second one from the isomorphism of ϵ^* for torsion sheaves from 2.5. Also we have

$$H(X, F) = H(X, F_A) \otimes \mathbb{C}$$

From the last two equations we can obtain, for example, the comparison of dimensions $H(X, F)$ and of $H_{et}(X_{et}, F_k^{et})$. One application given in BBD is to prove the vanishing of cohomology above degree n for affine X using the same result for etale cohomology.

5 Constructible coefficients

What happens if we replace Q_l with a constructible sheaf L? There are two main problems:

Our main tool was the comparison theorem for etale cohomology and analytic topology for torsion sheaf in 2.5. To take a non-torsion sheaf, we have to control the limit. As we already saw, for Q_l-sheaves, ϵ^* is not essentially surjective. Moreover, when passing from the field $\mathbb{C}$ to a finitly generated ring over $\mathbb{Z}$, we need a finiteness condition. The latter issue is more severe. In general

$$D_c^b(X_{\mathbb{C}}, F) \neq D_c^b(X_A, F)$$

for infinite objects like $A = \mathbb{Z}_l, Q_l, \ldots$. The way around this problem is explained in 6.1.8 of BBD.